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The Cyclizer Series of Infinite Permutation Groups

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The cyclizer series of infinite permutation groups

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Summary

The cyclizer of an infinite permutation group G is the group generated by the cycles involved in elements of G, along with G itself. There is an ascending subgroup series beginning with G, where each term in the series is the cyclizer of the previous term. We call this series the **cyclizer series** for G. If this series terminates then we say the **cyclizer length** of G is the length of the respective cyclizer series. We study several infinite permutation groups, and either determine their cyclizer series, or determine that the cyclizer series terminates and give the cyclizer length. In each of the infinite permutation groups studied, the cyclizer length is at most 3. We also study the structure of a group that arises as the cyclizer of the infinite cyclic group acting regularly on itself. Our study discovers an interesting infinite simple group, and a family of associated infinite characteristically simple groups.

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Chapter 1

Background material and Introduction

1.1 Elementary permutation group theory

Note: in this thesis we use the more widespread convention of writing functions on the left, rather than on the right as many group theorists prefer; thus for a set Ω and two functions $f, g : \Omega \to \Omega$ and $\omega \in \Omega$, $fg(\omega)$ means first apply g, then apply f.

Let Ω be a non-empty set. A **permutation** of Ω is a bijection of Ω onto itself. The set of all permutations of Ω form a group under composition of functions, called the **symmetric group** on Ω. We denote this group by $\mathrm{Sym}(\Omega)$. In the case where $\Omega = \{1, 2, \cdots, n\}$ for some natural number n we sometimes write S_n instead, and in the case where $\Omega = \mathbb{Z}$ we sometimes write S_∞ instead. We call a subgroup of $\mathrm{Sym}(\Omega)$ a **permutation group** on Ω. If G is a permutation group on Ω, then we say that n is the **degree** of G if $|\Omega| = n$.

Let G be a group and Ω be a non-empty set, and suppose there is a function $\mu : G \times \Omega \to \Omega$, $(g, \omega) = {}^g\omega$. We say that μ defines an **action** of G on Ω (or just that G **acts** on Ω) if:

- ${}^{id}\omega = \omega$ for all $\omega \in \Omega$
- ${}^h({}^g\omega) = {}^{gh}\omega$ for all $g, h \in G$, $\omega \in \Omega$.

If G acts on Ω, then we have a homomorphism $\varphi : G \to \mathrm{Sym}(\Omega)$, $\varphi(g) = \pi_g$, where $\pi_g : \Omega \to \Omega$, $\omega \mapsto {}^g\omega$. Therefore the image of G under φ is a permutation group, which we call the permutation group **induced** on Ω by G, and which we denote by G^{Ω}.

Suppose that G acts on Ω. Define a relation $\sim$ on Ω as follows: $\alpha \sim \beta$ if and only if $\beta = {}^g\alpha$ for some $g \in G$. The relation $\sim$ is an equivalence relation on Ω, and we call the equivalence classes the **orbits** of G. We say that G is **transitive** if there is only one orbit: in other words, G is transitive if for all $\alpha, \beta \in \Omega$, $\beta = {}^g\alpha$ for some $g \in G$.

Let G be an intransitive permutation group on Ω (that is, G is not transitive). Suppose that G has orbits Ω_i, for $i \in I$ where I is some index set for the set of orbits. Then G acts on each Ω_i, and thus induces a transitive permutation group G^{Ω_i} on Ω_i for each $i \in I$. We call these groups the **transitive constituents** of G.

Let X, Y be non-empty sets, let f be a function from X to Y, and let A be a non-empty subset of X, B be a non-empty subset of Y. The **image** of A is the set $\{f(a)|a \in A\}$, and we denote this by $f(A)$. In the context of our exposition, we can safely assume that no subset of A is an element of A, and so the definition of $f(A)$ is unambiguous. The **preimage** of B is the set $\{x \in X|f(x) \in B\}$, and we denote this by $f^{-1}(B)$. We denote the cardinality of the set X by $|X|$. If f is a bijection then $|f(A)| = |A|$ and $|f^{-1}(B)| = |B|$. If Z is another non-empty set and g is a function from Y to Z the **composition** of g and f is the function $g \circ f : X \to Z$, $(g \circ f)(x) = g(f(x))$ for all $x \in X$.

The infinite version of the **Dirichlet principle** states that if a countably infinite number of elements are to be put into a finite number of sets, then one set contains a countably infinite number of elements.

Let G be a group, and let H be a subgroup of G. The **normalizer** of H in G, denoted by $N_G(H)$ is

$$N_G(H) = \{g \in G : gHg^{-1} = H\}$$

It is straightforward to show that $H \trianglelefteq N_G(H)$, and that in fact $N_G(H)$ is the largest subgroup of G in which H is normal.

An **automorphism** of a group G is a group isomorphism $\varphi : G \to G$. A subgroup H of G is called **characteristic** in G if $\varphi(H) = H$ for every automorphism φ of G. A group G is called **characteristically simple** if it has no proper, nontrivial characteristic subgroups.

Proposition 1.1. *A direct product of isomorphic simple groups is characteristically simple.*

Proof. See Robinson, pg 85. [1] □

General references on group theory are Robinson [1], and Rotman [2]. General references on permutation groups are Dixon and Mortimer [3], and Cameron [4]; more information specifically focusing on infinite permutation groups can be found in Bhattacharjee, et al [5].

1.2 Elementary graph theory

A **graph** is a pair (V, E) where V is a non-empty set and E is set consisting of 2-element subsets of V. An element of V is called a **vertex**, and an element of E is called an **edge**. We do not allow more than one edge between any pair of distinct vertices, and we do not allow edge loops, with both ends of the edge being the same vertex. Two vertices u and v are said to be **adjacent** if $\{u, v\}$ is an edge.

A **path** from u to v is a sequence of vertices $u = v_0, v_1, \cdots, v_n = v$ such that $\{v_i, v_{i+1}\}$ is an edge for each $i \in \{0, 1, \cdots, n-1\}$. We say the **length** of a path $v_0, v_1, \cdots, v_n$ is the natural number n. The **distance** from u to v is the length of the shortest path from u to v. We say that a graph is **connected** if for all $u, v \in V$ there exists a path from u to v.

Let $\Gamma = (V, E)$ be a graph, and let f be a permutation of V. We say that f is **bounded** if the set of distances from v to $f(v)$, where v ranges over the whole of V, is bounded. We say that M is a **bound** for f if M is a bound for the set of distances from v to $f(v)$. Note that f is not required to be a graph automorphism but merely a permutation of the vertices.

A general reference on graph theory is Behzad, Chartrand and Lesniak-Foster [6].

1.3 The cyclizer function on finite permutation groups

Let $c : \Omega \to \Omega$ be a cycle in a permutation group acting on Ω. The **support** of c, denoted $\mathrm{supp}(c)$, is the set $\{x \in \Omega \mid c(x) \neq x\}$. Let G be a permutation group. We say that a cycle c is **involved** in $g \in G$ if $c^{-1}g$ fixes all points of the support of c.

Example 1.1. *The cycle* $(2, 3, 4)$ *is involved in the the permutation*

$$(1, 5)(2, 3, 4)(6, 7) \in S_7$$

and the cycle $(-35, -34)$ *is involved in the permutation*

$$\cdots (-3, -2)(-1, 0)(1, 2)(3, 4) \cdots \in S_\infty$$

The notion of a cycle c being involved in a permutation g is a generalisation of the notion that a cycle c is involved in the cycle decomposition of a finite permutation g. In fact these two notions are equivalent when G is a finite permutation group. Let G be a finite permutation group. The **cyclizer** of G, denoted $\mathrm{Cyc}(G)$, is the group generated by all the cycles involved in elements of G. We say that a group is **cycle-closed** if $\mathrm{Cyc}(G) = G$.

Example 1.2. *Let*

$$\begin{aligned} G = D_4 = \{id, (1, 2, 3, 4), (1, 3)(2, 4), (1, 4, 3, 2), \\ (1, 3), (2, 4), (1, 2)(3, 4), (1, 4)(2, 3) \end{aligned}$$

The set of cycles involved in elements of D_4 *is the set*

$$C = \{(1, 2), (1, 3), (1, 4), (2, 3), (2, 4), (3, 4), (1, 2, 3, 4), (1, 4, 3, 2)\}$$

and so $Cyc(G) = \langle C \rangle = S_4$*, since* C *contains all the* 2*-cycles of* S_4*.*

For a finite permutation group G of degree n we can define a series of groups

$$G = G_0 \leq G_1 \leq G_2 \leq \cdots$$

where $G_{i+1} = \mathrm{Cyc}(G_i)$ for every natural number i. We call this series the **cyclizer series** for G. This series must terminate in a finite number of steps, because each of the groups in the series is contained in the finite group S_n. Therefore there is $k \in \mathbb{N}$ such that $G_{k-1} \neq G_k = G_{k+1}$; we call this k the **length** of the cyclizer series for G.

It will often be necessary to refer to specific groups in the cyclizer series of a group G, so for every natural number k we define the groups $\mathrm{Cyc}^k(G)$ inductively, where $\mathrm{Cyc}^1(G) = \mathrm{Cyc}(G)$ and $\mathrm{Cyc}^{k+1}(G) = \mathrm{Cyc}(\mathrm{Cyc}^k(G))$. The cyclizer series for finite permutation groups was first considered by Cameron in [7]. Here, Cameron proves the following theorem:

Theorem 1.1 (Cameron)**.** *Let* G *be a permutation group on a finite set* X*. Then*

- G *is cycle-closed if and only if it is a direct product of symmetric groups and cyclic group of prime order,*
- $Cyc^3(G)$ *is cycle closed,*
- *There exists finite permutation groups* G *such that* $Cyc^2(G)$ *is not cycle closed; such a group, if transitive, is a* p*-group for some odd prime* p*.*

This result was expanded upon by Fiddes in [8]. Her work determined all finite groups with the maximal cyclizer series length of 3, and furthermore, established a near-complete classification of finite groups by cyclizer series length.

Theorem 1.2 (Fiddes)**.** *Let* G *be a transitive permutation group of a finite set* X*.*

- *If* G *has cyclizer length* 1*, then* G *has even order.*
- *If* G *has cyclizer length* 2*, then* G *has odd order, or* G *is a* 2*-group that does not have cyclizer length* 1*.*

- *G has cyclizer length 3 if and only if G is a particular subgroup of the iterated wreath product $\underbrace{C_p \; wr \; C_p \; wr \; \cdots \; wr \; C_p}_{n \text{ copies}}$, for some odd prime p.*

Theorem 1.3 (Fiddes)**.** *Let G be an intransitive permutation group of a finite set X, and let $G_1, G_2, \cdots G_k$ be the transitive constituents of G. Then*

$$Cyc(G) = Cyc(G_1) \times Cyc(G_2) \times \cdots \times Cyc(G_k)$$

and hence G is cycle-closed if and only if all its transitive constituents are. Furthermore, the cyclizer length of G is the maximum of the cyclizer lengths of its transitive constituents.

The specifics of Theorem 1.2, particularly the groups with cyclizer length 3, are rather technical and so we refer the reader to Fiddes and Smith [9] for a detailed and focused treatment of this result. For an application of Cameron's result in Theorem 1.1, the author refers the reader to a paper by Lenart and Ray on Hopf Algebras [**?**].

1.4 The cyclizer function on infinite permutation groups

Let Ω be an infinite set. We say that a permutation of Ω is **finitary** if its support is a finite set. The set of all finitary permutations of Ω form a group, denoted FS(Ω). The subgroup of $FS(\Omega)$ consisting of all the even finitary permutation of Ω is called the **finitary alternating group** on Ω, and is denoted Alt(Ω).

Observe that if G is an infinite permutation group, then G is not necessarily contained in the group generated by all the cycles involved in elements of G.

Example 1.3. *Let $\pi = \cdots(-3,-2)(-1,0)(1,2)(3,4)\cdots \in S_\infty$, and let $G = \langle \pi \rangle$. The group generated by the cycles involved in elements of G is a subgroup of $FS(\mathbb{Z})$, which does not contain the permutation π since it has infinite support.*

Defining the cyclizer of an infinite permutation group is a slightly different process than in the finite permutation group case. Cameron [7] suggests several different ways this can be done, but in this thesis we concern ourselves with the following generalisation of his:

Definition 1.1. *Let G be a permutation group. The cyclizer of G, denoted $Cyc(G)$, is the group generated by G and all the cycles involved in elements of G.*

Notice that if G is finite then this definition agrees with our earlier notion of the cyclizer of G, but avoids the problem highlighted in Example 1.3 when G is infinite. As in the finite case, define the **cyclizer series** of an infinite permutation group G to be the series

$$G = G_0 \leq G_1 \leq G_2 \leq \cdots$$

where $G_{i+1} = \mathrm{Cyc}(G_i)$ for every natural number i. Unlike in the finite case, there is no a priori reason why this series must terminate; if it does terminate, i.e. if there exists $k \in \mathbb{N}$ such that $G_{k-1} \neq G_k = G_{k+1}$, then we call this k the **length** of the cyclizer series for G. It is not known if there is a maximal length for the cyclizer series of an infinite permutation group. In [7] Cameron conjectures that 3 is the maximal series length. The groups studied in Chapters 3 and 5 all have cyclizer length ≤ 3, but we are so far unable to provide a proof of Cameron's conjecture.

As in the finite case, for every natural number k we define the groups $\mathrm{Cyc}^k(G)$ inductively, where $\mathrm{Cyc}^1(G) = \mathrm{Cyc}(G)$ and $\mathrm{Cyc}^{k+1}(G) = \mathrm{Cyc}(\mathrm{Cyc}^k(G))$. The known results about the cyclizer series of infinite permutation groups are presented below:

Proposition 1.2 (Cameron)**.** *Let G be a transitive permutation group in which all the cycles of all elements of G are finite. Then $Cyc(G) \leq FS(\Omega).G$ Hence $Cyc^3(G) = FS(\Omega).G$ is cycle-closed.*

The proof of Proposition 1.2 can be found in Cameron [7]. Before stating the next result, we need a definition from Fiddes's thesis:

Definition 1.2. *A permutation π of the integers is said to be* **modular** *(or* **modularly defined***) if there is a natural number n such that for each integer z, $\pi(z) = z + \mu_z$, where μ_z depends only on the congruence class of z modulo n.*

Any such n is called a modulus for π, and the smallest such is called the principal modulus of π.

Example 1.4. *The permutation $f : \mathbb{Z} \to \mathbb{Z}$,*

$$f(x) = \begin{cases} x+2 & x \equiv 0 \ mod\ 3 \\ x-1 & x \equiv 1,2 \ mod\ 3 \end{cases}$$

is modular, with modulus 3 and $\mu_0 = 2$, $\mu_1 = \mu_2 = -1$.

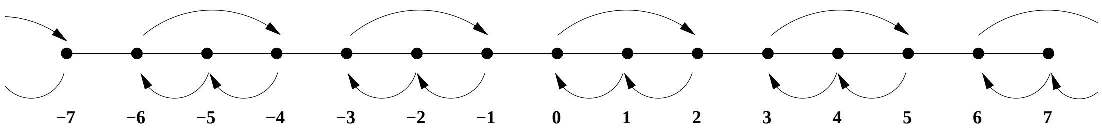

Figure 1-1: Geometric interpretation of the modular permutation f

The geometric interpretation of f seen in Figure 1-1 is a handy way to understand its overall structure, and so we will be interpreting permutations of Ω as permutations of a graph Γ whose vertex set is Ω many times throughout this work. Now we state the known results about the cyclizer series of some specific infinite permutation groups.

Theorem 1.4 (Cameron)**.** *Let G be the permutation group induced by the infinite cyclic group $\mathbb{Z}$ acting on itself in the natural way.*

- *$Cyc(G)$ is the group of modular permutations of $\mathbb{Z}$,*
- *$Cyc^2(G)$ is the semidirect product $FS(\mathbb{Z}) \rtimes Cyc(G)$,*
- *$Cyc^3(G)$ is the set of all permutation g of $\mathbb{Z}$ for which there exist $r > 0$ and $h_+, h_- \in Cyc(G)$ such that $g(x) = h_+(x)$ for $x > r$ and $g(x) = h_-(x)$ for $x < -r$*
- *$Cyc^4(G) = Cyc^3(G)$, in other words, $Cyc^3(G)$ is cycle-closed.*

Again, the proof of Theorem 1.4 can be found in Cameron [7]; a slightly different proof may be found in Fiddes' thesis [8].

Theorem 1.5 (Fiddes). *Let D be the permutation group induced by the infinite dihedral group acting on $\mathbb{Z}$ in the natural way, and let G be the permutation group induced by $\mathbb{Z}$ acting on itself in the natural way.*

- $Cyc(D) = Cyc^2(G) \cup Cyc^2(G)\tau$, *where* $\tau : \mathbb{Z} \to \mathbb{Z}$, $\tau(z) = -z$ *for all* $z \in \mathbb{Z}$
- $Cyc^2(D)$ *is cycle-closed.*

The proof of Theorem 1.5 can be found in [8].

This thesis is concerned with investigating some interesting infinite permutation groups, and determining information about their respective cyclizer series. In Chapter 2 we investigate the group of modular permutations of the integers, that appears in the cyclizer series of the infinite cyclizer group. We recall what it means for a permutation of the integers to have finite flow (a concept that first appeared in Fiddes' thesis [8]) and observe that all of the modular permutations have this property. We then look at the subgroup of modular permutations which have zero flow, and conclude with Theorem 2.3, the main result of the chapter, which shows that this group is simple.

In Chapter 3 we investigate the cyclizer series of a family of infinite permutation groups G_n for each natural number $n \geq 3$, that in some sense generalise the infinite cyclic group. For each natural number $n \geq 3$ the group G_n arises as a permutation group on a type of graph called an n-branched star, which we define in section 3.1. We discover that, modulo the finitary permutations of the underlying graph, the group G_n is isomorphic to $n-1$ copies of the infinite cyclic group (Lemma 3.2). The cyclizer series of G_n is expectedly related to the cyclizer series of the infinite cyclic group that Cameron and Fiddes have already determined, as we show when we determine $\mathrm{Cyc}(G_n)$ in Lemmas 3.4 and 3.5. This leads into our final important result of the chapter in Theorem 3.3, that the cyclizer length of each G_n is just 1.

Chapter 4 deals with the generalising the concept of finite flow to permutations of the vertices of an n-branched star. We investigate the zero flow subgroup of $\mathrm{Cyc}(G_n)$, the main result being Theorem 4.2 which tells us that, modulo the finitary permutations of the underlying graph, this subgroup is a characteristically simple group. We end the chapter with an investigation of the normaliser of the finite flow permutations of an n-branched star, with the main result being Theorem 4.3, that the normaliser is a semi-direct product of the finite flow permutations with S_n.

Chapter 5 investigates the cyclizer series of two infinite permutation groups. The first of these, the cross group G_+, is a permutation group of a 4-branched star that is a subgroup of the group G_4 studied in Chapter 3. Our main result for the cross group is Theorem 5.1, which states that G_+ has cyclizer length at most 2. The second group we investigate is the ladder group G_L, so called because it is a permutation group of the vertices of a graph which resembles an infinitely long ladder. With Theorem 5.2 we prove that G_L has a cyclizer length of 2.

Chapter 6 contains some open questions and conjectures that arise from the thesis, and how the work in Chapter 5 can be used to help fully understand the cyclizer series of $\mathbb{Z}_n$ acting regularly.

Chapter 2

Modular permutations of the integers

In this chapter we investigate the group of modular permutations of $\mathbb{Z}$. Our ultimate aim is to prove that a particular subgroup of the modular permutations is a simple group. We begin with a discussion of a group that contains the modular permutations of $\mathbb{Z}$: the finite flow permutations of $\mathbb{Z}$.

2.1 Finite flow permutations of the integers

In [8], Fiddes introduces a class of permutations of the integers; the finite flow permutations. We define these below.

Definition 2.1. *Let $g \in Sym(\mathbb{Z})$ and let $x \in \mathbb{Z} + \frac{1}{2}$. Define*

$$\phi_x^+(g) = |\{z \in \mathbb{Z} : z < x, f(z) > x\}|$$

$$\phi_x^-(g) = |\{z \in \mathbb{Z} : z > x, f(z) < x\}|$$

We say g has **finite flow** *(or that g is a* **finite flow permutation***) if $\phi_x^+(g)$ and $\phi_x^-(g)$ are finite for all $x \in \mathbb{Z} + \frac{1}{2}$. If g has finite flow, we define $\phi_x(g) := \phi_x^+(g) - \phi_x^-(g)$. We call $\phi_x(g)$ the* **net flow** *of a finite flow permutation g.*

Example 2.1. *The identity permutation $id_{\mathbb{Z}}$ has finite flow: The quantities $\phi_x^+(id_{\mathbb{Z}})$ and $\phi_x^-(id_{\mathbb{Z}})$ are both equal to 0, for all $x \in \mathbb{Z} + \frac{1}{2}$, and so the net flow, $\phi_x(id_{\mathbb{Z}})$, is also equal to 0 for all $x \in \mathbb{Z}$.*

The permutation $\sigma : \mathbb{Z} \to \mathbb{Z}$, $\sigma(z) = z + 1$ *for all* $z \in \mathbb{Z}$, *has finite flow and its net flow is* 1, *since* $\phi_x^+(\sigma) = 1$ *and* $\phi_x^-(\sigma) = 0$ *for all* $x \in \mathbb{Z}$.

The permutation $\tau : \mathbb{Z} \to \mathbb{Z}$, $\tau(z) = -z$ *for all* $z \in \mathbb{Z}$, *is not a finite flow permutation, as the sets* $\{z \in \mathbb{Z} : z < \frac{1}{2}, \tau(z) > \frac{1}{2}\}$ *and* $\{z \in \mathbb{Z} : z < \frac{1}{2}, \tau(z) > \frac{1}{2}\}$ *are both infinite.*

There is a very natural geometric understanding of a finite flow permutation. For example, consider a geometric interpretation of the permutation σ from Example 2.1: In this interpretation, the quantity $\phi^+_{\frac{1}{2}}(\sigma)$ essentially counts the num-

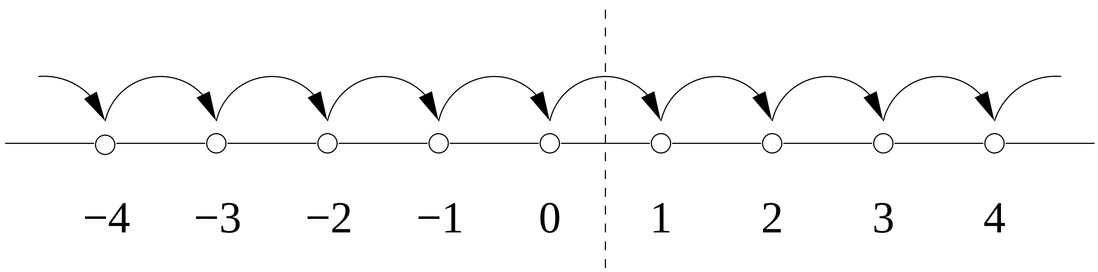

Figure 2-1: Geometric interpretation of the map σ

ber of right-facing arrows that cross the vertical dotted line. Similarly, $\phi^-_{\frac{1}{2}}(\sigma)$ counts the number of left-facing arrows that cross the vertical line. Since both these numbers are finite for σ, the net flow of σ exists, and in this interpretation it essentially counts the net difference between the right-facing and left-facing arrows that cross the vertical line.

It turns out that the set of all finite flow permutations of the integers form a subgroup of $\mathrm{Sym}(\mathbb{Z})$. This was originally thought to be shown in Fiddes' thesis [8], but unfortunately one of the theorems used to do so is flawed. The theorem in question is Theorem 7.4, which appears on page 77 of Fiddes' thesis and states that if $f, g \in \mathrm{Sym}(\mathbb{Z})$ have finite flow, then $\phi_x^+(g \circ f) = \phi_x^+(f) + \phi_x^+(g)$ for all $x \in \mathbb{Z} + \frac{1}{2}$. We see that this is false with the following example:

Example 2.2. *Let*

$$f = (\cdots, -2, -1, 0, 1, 2, \cdots)$$

and

$$g = (\cdots, 4, 2, 0, -2, -4, \cdots) \circ (\cdots, 5, 3, 1, -1, -3 \cdots)$$

Then f and g have finite flow, because $\phi_x^+(f) = 1$, $\phi_x^-(f) = 0$ for all $x \in \mathbb{Z} + \frac{1}{2}$, and $\phi_x^+(g) = 0$, $\phi_x^-(g) = 2$ for all $x \in \mathbb{Z} + \frac{1}{2}$. Now, since

$$g \circ f = (\cdots, 2, 1, 0, -1, -2, \cdots)$$

it follows that $\phi_x^+(g \circ f) = 0$ for all $x \in \mathbb{Z} + \frac{1}{2}$. But $\phi_x^+(f) + \phi_x^+(g) = 1$, and so we have a counter-example.

Although the statement $\phi_x^+(g \circ f) = \phi_x^+(f) + \phi_x^+(g)$ for all $x \in \mathbb{Z} + \frac{1}{2}$ is false, it is true that composing finite flow permutations sums their net flow. This similar result allows us to correctly prove that the finite flow permutations form a subgroup of $\mathrm{Sym}(\mathbb{Z})$; for the sake of logical consistency, we prove this in full. We begin by proving the statement about net flow.

Lemma 2.1. *Let f, g be finite flow permutations of the integers. Then $g \circ f$ has finite flow, and in particular for all $x \in \mathbb{Z} + \frac{1}{2}$*

$$\phi_x(g \circ f) = \phi_x(f) + \phi_x(g)$$

Proof. Let $x \in \mathbb{Z} + \frac{1}{2}$ and define the following subsets of the integers:

$$A = \{z < x : g(z) > x\}, B = \{z < x : g(z) < x\}$$
$$C = \{z > x : g(z) < x\}, D = \{z > x : g(z) > x\}$$

Consider

$$\phi_x(g \circ f) = |\{z < x : (g \circ f)(z) > x\}| - |\{z > x : (g \circ f)(z) < x\}|$$

We can write

$$\{z < x : (g \circ f)(z) > x\} = (f^{-1}(A) \cap (-\infty, x)) \cup (f^{-1}(D) \cap (-\infty, x))$$

where $(-\infty, x)$ is the interval $\{z \in \mathbb{Z} : z < x\}$ and the union is disjoint. The set $f^{-1}(A) \cap (-\infty, x)$ is finite because A is finite due to g having finite flow, and $|A| = |f^{-1}(A)|$ because f is a bijection. The set $f^{-1}(D) \cap (-\infty, x)$ is also finite because it is a subset of $\{z < x : f(z) > x\}$, which is finite because f has finite flow. Consequently, $\{z < x : (g \circ f)(z) > x\}$ is finite. Also, we can write

$$\{z > x : (g \circ f)(z) < x\} = (f^{-1}(B) \cap (x, \infty)) \cup (f^{-1}(C) \cap (x, \infty))$$

where (x, ∞) is the interval $\{z \in \mathbb{Z} : z > x\}$ and the union is again disjoint. The set $f^{-1}(B) \cap (x, \infty)$ is finite because it is a subset of $\{z > x : f(z) < x\}$, which is finite because f has finite flow. The set $f^{-1}(C) \cap (x, \infty)$ is also finite, because C is finite due to g having finite flow and $|C| = |f^{-1}(C)|$ because f is a bijection. Hence $\{z > x : (g \circ f)(z) < x\}$ is finite, therefore $g \circ f$ has finite flow and

$$\begin{aligned} \phi_x(g \circ f) = |f^{-1}(A) \cap (-\infty, x)| &+ |f^{-1}(D) \cap (-\infty, x)| \\ &- |f^{-1}(B) \cap (x, \infty)| - |f^{-1}(C) \cap (x, \infty)| \end{aligned}$$

To conclude we show that $\phi_x(g \circ f) = \phi_x(f) + \phi_x(g)$.

It follows from the earlier definitions that $\phi_x(g) = |A| - |C|$. Consider

$$\phi_x(f) = |\{z < x : f(z) > x\}| - |\{z > x : f(z) < x\}|$$

It follows that

$$\{z < x : f(z) > x\} = (f^{-1}(C) \cap (-\infty, x)) \cup (f^{-1}(D) \cap (-\infty, x))$$

where the union is disjoint. Furthermore,

$$\{z > x : f(z) < x\} = (f^{-1}(A) \cap (x, \infty)) \cup (f^{-1}(B) \cap (x, \infty))$$

where again the union is disjoint. Thus

$$\begin{aligned} \phi_x(f) + \phi_x(g) = |f^{-1}(C) \cap (-\infty, x)| &+ |f^{-1}(D) \cap (-\infty, x)| \\ &- |f^{-1}(A) \cap (x, \infty)| - |f^{-1}(B) \cap (x, \infty)| + |A| - |C| \end{aligned}$$

Since f is a bijection, it follows that

$$|A| = |f^{-1}(A)| = |f^{-1}(A) \cap (-\infty, x)| + |f^{-1}(A) \cap (x, \infty)|$$
$$|C| = |f^{-1}(C)| = |f^{-1}(C) \cap (-\infty, x)| + |f^{-1}(C) \cap (x, \infty)|$$

and so

$$\begin{aligned}\phi_x(f) + \phi_x(g) = |f^{-1}(A) \cap (-\infty, x)| &+ |f^{-1}(D) \cap (-\infty, x)| \\ &- |f^{-1}(B) \cap (x, \infty)| - |f^{-1}(C) \cap (x, \infty)|\end{aligned}$$

which is equal to $\phi_x(g \circ f)$. □

Let G_{fin} denote the set of permutations of $\mathbb{Z}$ that have finite flow.

Theorem 2.1. $G_{\text{fin}} \leq Sym(\mathbb{Z})$

Proof. Example 2.1 shows that $id_{\mathbb{Z}}$ has finite flow. If f, g have finite flow then Lemma 2.1 tells us that $g \circ f$ has finite flow, and so G is closed under composition. Finally, it is easy to to check that $\phi_x^+(g^{-1}) = \phi_x^-(g)$ and $\phi_x^-(g^{-1}) = \phi_x^+(g)$, hence $\phi_x(g^{-1}) = -\phi_x(g)$ for all $x \in \mathbb{Z} + \frac{1}{2}$ and thus g^{-1} has finite flow. □

There are some other elementary observations we can make regarding finite flow permutations, that are both useful and illuminating. We begin with the observation originally made by Fiddes, that the net flow of a finite flow permutation at $x \in \mathbb{Z} + \frac{1}{2}$ is independent of the choice of x.

Lemma 2.2. *Let $g \in G_{fin}$ and assume that $\phi_x(g) = r$ for some $x \in \mathbb{Z} + \frac{1}{2}$. Then $\phi_y(g) = r$ for all $y \in \mathbb{Z} + \frac{1}{2}$.*

Proof. Fiddes thesis, Theorem 7.3 [8] □

It follows from Lemma 2.2 that we can define the net flow of a finite flow permutation g to be the net flow of g at any $x \in \mathbb{Z} + \frac{1}{2}$. Furthermore we can define a function ϕ, that takes a finite flow permutation and gives us its net flow. In other words. we have $\phi : G_{\text{fin}} \to \mathbb{Z}, \phi(g) = \phi_x(g)$ for some $x \in \mathbb{Z} + \frac{1}{2}$. Theorem 2.1 and Lemma 2.1 together imply that ϕ is a group homomorphism, and we can also see that ϕ is surjective: for each $z \in \mathbb{Z}$, the permutation $g : \mathbb{Z} \to \mathbb{Z}, g(a) = a + z$ has $\phi(g) = z$. We will return to this function in section 2.2.

Another interesting observation is that, when determining if a given permutation has finite flow, it is sufficient to check that $\phi_x^+(f)$, $\phi_y^-(f)$ are finite for any $x, y \in \mathbb{Z} + \frac{1}{2}$.

Lemma 2.3. *Let $f \in Sym(\mathbb{Z})$ and assume that $\phi_x^+(f)$ and $\phi_y^-(f)$ are both finite. Then f has finite flow.*

Proof. Let $a \in \mathbb{Z} + \frac{1}{2}$ and assume that $\phi_a^+(f) = \infty$ (therefore $a \neq x$). There are two cases to consider.

Case 1 **-** $a > x$**:** Since $\phi_a^+(f) = \infty$, there are infinitely many integers $z < a$ such that $f(z) > a > x$. Infinitely many of these integers must also satisfy $z < x$. However, this contradicts the finiteness of $\phi_x^+(f)$.

Case 2 **-** $a < x$**:** Since $\phi_a^+(f) = \infty$, there are infinitely many integers $z < a < x$ such that $f(z) > a$. Infinitely many of these integers must also satisfy $f(z) > x$, because f is a bijection. However, this contradicts the finiteness of $\phi_x^+(f)$.

Therefore our assumption that $\phi_a^+(f) = \infty$ was flawed and so $\phi_a^+(f)$ is finite. Now assume that $\phi_a^-(f) = \infty$ (therefore $a \neq y$). Again we have two cases to consider.

Case 1 **-** $a > y$**:** Since $\phi_a^-(f) = \infty$, there are infinitely many integers $z > a > y$ such that $f(z) < a$. Infinitely many of these integers must also satisfy $f(z) < y$, because f is a bijection. However, this contradicts the finiteness of $\phi_y^-(f)$.

Case 2 **-** $a < y$**:** Since $\phi_a^-(f) = \infty$, there are infinitely many integers $z > a$ such that $f(z) < a < y$. Infinitely many of these integers must also satisfy $z > y$. However, this contradicts the finiteness of $\phi_y^-(f)$.

Therefore our assumption that $\phi_a^-(f) = \infty$ was flawed and so $\phi_a^-(f)$ is finite. Since both of $\phi_a^+(f)$ and $\phi_a^-(f)$ are finite, and $a \in \mathbb{Z} + \frac{1}{2}$ was chosen arbitrarily, f has finite flow. ∎

Since by definition a finite flow permutation f has $\phi_x^+(f)$, $\phi_y^-(f)$ finite for any

$x, y \in \mathbb{Z} + \frac{1}{2}$, Lemma 2.3 allows us to make an equivalent definition of finite flow.

Definition 2.2. *Let $f \in Sym(\mathbb{Z})$. We say that f has* **finite flow** *if $\phi_x^+(f)$, $\phi_y^-(f)$ are finite for some $x, y \in \mathbb{Z} + \frac{1}{2}$.*

In particular, a permutation g of $\mathbb{Z}$ has finite flow if and only if the quantities $\phi^+_{-\frac{1}{2}}(g)$, $\phi^-_{\frac{1}{2}}(g)$ are finite. These values are the sizes of the sets $\{z < 0 : g(z) \geq 0\}$ and $\{z > 0 : g(z) \leq 0\}$ respectively, which can also be written as $\mathbb{Z}^- \setminus g^{-1}(\mathbb{Z}^-)$ and $\mathbb{Z}^+ \setminus g^{-1}(\mathbb{Z}^+)$ respectively, where

$$\mathbb{Z}^+ := \{z \in \mathbb{Z} : z > 0\},\ \mathbb{Z}^- := \{z \in \mathbb{Z} : z < 0\}$$

Since g is a bijection, it follows that $|g(X)| = |X|$ for any subset X of $\mathbb{Z}$, and so

$$\phi^+_{-\frac{1}{2}}(g) = |\{z < 0 : g(z) \geq 0\}| = |\mathbb{Z}^- \setminus g^{-1}(\mathbb{Z}^-)| = |g(\mathbb{Z}^-) \setminus \mathbb{Z}^-|$$

$$\phi^-_{\frac{1}{2}}(g) = |\{z > 0 : g(z) \leq 0\}| = |\mathbb{Z}^+ \setminus g^{-1}(\mathbb{Z}^+)| = |g(\mathbb{Z}^+) \setminus \mathbb{Z}^+|$$

As a result, we have another equivalent definition of a finite flow permutation.

Definition 2.3. *Let $g \in Sym(\mathbb{Z})$. We say that g has* **finite flow** *if the sets $g(\mathbb{Z}^+) \setminus \mathbb{Z}^+$, $g(\mathbb{Z}^-) \setminus \mathbb{Z}^-$ are finite.*

The advantage of Definition 2.3 is that proving G_{fin} is closed under composition can now be done much more swiftly: Let $f, g \in G_{\text{fin}}$. Since $f \in G_{\text{fin}}$ it follows that $f(\mathbb{Z}^+) \setminus \mathbb{Z}^+$ is a finite set. Since g is a bijection,

$$|f(\mathbb{Z}^+) \setminus \mathbb{Z}^+| = |g(f(\mathbb{Z}^+) \setminus \mathbb{Z}^+)| = |(g \circ f)(\mathbb{Z}^+) \setminus g(\mathbb{Z}^+)|$$

and so $(g \circ f)(\mathbb{Z}^+) \setminus g(\mathbb{Z}^+)$ is a finite set. Furthermore $g \in G_{\text{fin}}$ implies that $g(\mathbb{Z}^+) \setminus \mathbb{Z}^+$ is finite, so $\mathbb{Z}^+$ and $g(\mathbb{Z}^+)$ differ by only a finite number of elements, and so $(g \circ f)(\mathbb{Z}^+) \setminus \mathbb{Z}^+$ is a finite set. Similarly, $(g \circ f)(\mathbb{Z}^-) \setminus \mathbb{Z}^-$ is a finite set and so $g \circ f$ has finite flow. The disadvantage of this slick method is we do not see that composing finite flow permutations sums their net flow, and so we miss out on the homomorphism ϕ defined earlier.

Let c be an infinite cycle and $z \in \operatorname{supp}(c)$. We say that c is **positive** if

$\lim\limits_{n\to\pm\infty} c^n(z) = \pm\infty$ and c is **negative** if $\lim\limits_{n\to\pm\infty} c^n(z) = \mp\infty$. We say that c is **positively aligned** if $\lim\limits_{n\to\pm\infty} c^n(z) = \infty$ and c is **negatively aligned** if $\lim\limits_{n\to\pm\infty} = -\infty$. Observe that a positive cycle has flow 1, a negative cycle has flow -1, and an aligned cycle has flow 0. If c is an infinite cycle involved in a finite flow permutation f, then c must take one of the four forms above. Any finite cycle has zero flow, and so $\phi_x^+(f)$ is the number of positive cycles involved in f, and $\phi_x^-(f)$ is the number of negative cycles involved in f.

Observe that a modular permutation has finite flow, because for a modular permutation π, the set

$$\{\pi(z) - z | z \in \mathbb{Z}\}$$

is finite, and therefore bounded.

2.2 Elementary properties of moduar permutations

Modular permutations of $\mathbb{Z}$ were originally defined in Chapter 1, but there is an alternative definition that will be handy to use at times.

Definition 2.4. *Let $f \in Sym(\mathbb{Z})$. We say that f is* **modular** *if there exists $n \in \mathbb{N}$ such that for all $x \in \mathbb{Z}$,*

$$f(x+n) = f(x) + n$$

We call n a **modulus** *of f, and denote the set of all modular permutations by G_{mod}.*

We will shortly see the equivalence of the two competing definitions of modularity, but first we deduce some straightforward consequences of Definition 2.4.

Lemma 2.4. *Let $f \in Sym(\mathbb{Z})$ be modular with modulus n. Then for all $r, x \in \mathbb{Z}$,*

$$f(x+rn) = f(x) + rn$$

Proof. Let $x \in \mathbb{Z}$. Firstly, consider the case where $r \in \mathbb{N}$. We prove this by induction on r. When $r = 1$ there is nothing to prove, as this is the definition of a modular permutation. If the statement holds for $r = m$, then

$$f(x+(m+1)n) = f((x+n)+mn) = f(x+n)+mn = f(x)+n+mn = f(x)+(m+1)n$$

by the induction hypothesis and the definition of a modular permutation. Therefore the statement is true for $r = m+1$ and hence true for all $r \in \mathbb{N}$ by induction.

Now consider the case where $r \leq 0$. When $r = 0$ the statement is trivially true. For $r < 0$, write $r = -s$ for $s \in \mathbb{N}$ and instead prove $f(x - sn) = f(x) - sn$ for all $x \in \mathbb{Z}$ by induction on s. When $s = 1$, let $x \in \mathbb{Z}$ and let $y = x - n$ for $y \in \mathbb{Z}$. Then

$$f(x) - n = f(y + n) - n = f(y) + n - n = f(y) = f(x - n)$$

by the definition of a modular permutation. Hence the statement is true for $s = 1$. If the statement is true for $s = m$, then

$$f(x-(m+1)n) = f((x-n)-mn) = f(x-n)-mn = f(x)-n-mn = f(x)-(m+1)n$$

by the induction hypothesis and the definition of a modular permutation. Therefore the statement is true for $s = m+1$ and hence true for all $s \in \mathbb{N}$ by induction, which completes the proof. $\square$

Lemma 2.5. *Let f be a modular permutation with modulus n. Then for all $k \in \mathbb{Z}$,*

$$f^k(x + n) = f^k(x) + n$$

for all $x \in \mathbb{Z}$.

Proof. We begin with the case of $k \in \mathbb{N}$, which we prove by induction. The case $k = 1$ is just the definition of modular, so the statment is true for $k = 1$. If the statement holds for $k = m$ then

$$f^{m+1}(x + n) = f(f^m(x + n)) = f(f^m(x) + n) = f(f^m(x)) + n = f^{m+1}(x) + n$$

by the induction hypothesis and the definition of a modular permutation. So the statement holds for $k = m + 1$, hence the statement holds for all $k \in \mathbb{N}$ by induction.

When $k = 0$ there is nothing so prove, so it remains to deal with the case $k \in \mathbb{Z}$, $k < 0$. Notice that if f is modular with modulus n, then f^{-1} is also modular with modulus n; for all $x \in \mathbb{Z}$, write $x = f(y)$ for some $y \in \mathbb{Z}$, then since f is modular,

$$f^{-1}(x+n) = f^{-1}(f(y)+n) = f^{-1}(f(y+n)) = y+n = f^{-1}(x)+n$$

This implies that

$$f^{-l}(x+n) = (f^{-1})^l(x+n) = (f^{-1})^l(x)+n = f^{-l}(x)+n$$

for all $x \in \mathbb{Z}$ and for all $l \in \mathbb{N}$ by the previous paragraph. Let $k \in \mathbb{Z}, k < 0$ and write $k = -l$ for $l \in \mathbb{N}$. Then

$$\begin{aligned} f^k(x+n) &= f^{-l}(x+n) \\ &= f^{-l}(x)+n \\ &= f^k(x)+n \end{aligned}$$

for all $x \in \mathbb{Z}$ and so the result holds for all $k \in \mathbb{Z}, k < 0$. Hence the result holds for all $k \in \mathbb{Z}$. □

Corollary 2.1. *Let f be a modular permutation with modulus n. Then for all $k, r \in \mathbb{Z}$,*

$$f^k(x+rn) = f^k(x)+rn$$

for all $x \in \mathbb{Z}$.

Proof. Lemmas 2.4 and 2.5. □

We already know that G_{mod} is a group, because Cameron (and later Fiddes) showed that it is the cyclizer of the infinite cyclic group acting regularly. However, we can show this directly.

Proposition 2.1. *G_{mod} is a group.*

Proof. The identity permutation is modular with modulus 1, since

$$id_{\mathbb{Z}}(x+1) = x+1 = id_{\mathbb{Z}}(x)+1$$

for all $x \in \mathbb{Z}$. We proved in Lemma 2.5 that if f is modular with modulus n, then f^{-1} is also modular with modulus n, and so $f^{-1} \in G_{\text{mod}}$. Finally, if f and g are modular permutations with moduli n and p then fg is modular with modulus np; for all $x \in \mathbb{Z}$,

$$fg(x+np) = f(g(x)+np) = f(g(x)+pn) = fg(x)+np$$

by two applications of Lemma 2.4. $\square$

Now we show the equivalence of the two definitions of a modular permutation. Let f be a permutation of $\mathbb{Z}$. Suppose that f satisfies the conditions of Definition 1.2, i.e. for each $z \in \mathbb{Z}$ we have $f(z) = z + \mu_z$ where μ_z depends only on the congruence class of z modulo n. Let $x \in \mathbb{Z}$ and assume that $x \equiv a \bmod n$ (i.e. $f(x) = x + \mu_a$). It follows that $x + n \equiv a \bmod n$, and so

$$f(x+n) = x+n+\mu_a = (x+\mu_a)+n = f(x)+n$$

Therefore f satisfies the conditions of Definition 2.4.

Conversely, suppose that f satisfies the conditions of Definition 2.4, i.e. there is an $n \in \mathbb{N}$ such that $f(x+n) = f(x)+n$ for all $x \in \mathbb{Z}$. Let $z \in \mathbb{Z}$, assume that $z \equiv a \bmod n$ and let $\mu_a = f(a) - a$. Since $z \equiv a \bmod n$, we can write $z = a + rn$ for some $r \in \mathbb{Z}$. Using Lemma 2.4 we see that

$$f(z) = f(a+rn) = f(a)+rn = a+\mu_a+rn = (a+rn)+\mu_a = z+\mu_a$$

and so f satisfies the conditions of Definition 1.2. Hence the two definitions are equivalent. In Johnson [10], he defines an 'n-periodic permutation' of $\mathbb{Z}$. This definition is identical to our alternative definition of a modular permutation with modulus n, Definition 2.4. Furthermore Johnson observes that the group of permutations which are n-periodic for some $n \in \mathbb{N}$, which he denotes $\mathrm{Sym}_*(\mathbb{Z})$, contains the group of modular permutations G_{mod}, and asks whether this containment is proper or not. Since we have observed that an n-periodic permutation and a modular permutation with modulus n are the same thing, clearly $\mathrm{Sym}_*(\mathbb{Z}) = G_{\mathrm{mod}}$ and so Johnson's question is answered in the negative.

There is a very natural generating set for G_{mod}. Since G_{mod} is the cyclizer of the infinite cyclic group C_∞ acting naturally, it is generated by the set of all cycles involved in the elements of the induced permutation group $C_\infty^{\mathbb{Z}} \leq \mathrm{Sym}(\mathbb{Z})$. These cycles are the elementary permutations, introduced by Fiddes in [8].

Definition 2.5. *Let $g \in Sym(\mathbb{Z})$. We say that g is an* **elementary permutation** *if there is $i \in \mathbb{N}$ and $j \in \{1, 2, \cdots, i\}$ such that*

$$g(x) = x + i, \;\; \textit{if } x \equiv j \; \textit{(mod) } i$$

and g fixes everything else.

Since any modulus of a permutation $f \in G_{\mathrm{mod}}$ is a natural number, there exists a smallest modulus of f; we shall call this the **principal modulus** of f. Every modulus of f is divisible by the prinicpal modulus of f, as we shall see below.

Lemma 2.6. *Let $f \in G_{mod}$ and let n_0 be the principal modulus of f. If n is any modulus of f, then n_0 divides n.*

Proof. Let $n = kn_0 + l$ for some $k \in \mathbb{N}$ and $l \in \{0, 1, 2, \cdots, n_0 - 1\}$. Let $x \in \mathbb{Z}$. Since n is a modulus of f, we know that $f(x + n) = f(x) + n$. Also, by Lemma 2.4,

$$f(x + n) = f(x + (kn_0 + l)) = f((x + l) + kn_0) = f(x + l) + kn_0$$

and so $f(x) + n = f(x + l) + kn_0$. Hence

$$f(x + l) - f(x) = n - kn_0 = l$$

and so $f(x + l) = f(x) + l$ for all $x \in \mathbb{Z}$. If $l \neq 0$, then by definition, f has modulus l. But $l < n_0$, so this would contradict the minimality of n_0, hence $l = 0$ and so n_0 divides n. $\square$

Whilst it may seem that G_{mod} would depend on the fact that we are implicitly using the natural ordering of $\mathbb{Z}$, it turns out not to be the case. Any reordering of $\mathbb{Z}$ can also be thought of as a permutation of $\mathbb{Z}$, in other words an element of $\text{Sym}(\mathbb{Z})$. For any fixed $f \in \text{Sym}(\mathbb{Z})$ we say that $g \in \text{Sym}(\mathbb{Z})$ is **modular with respect to** f if the conjugate $g^f \in G_{\text{mod}}$, and denote the set of all modular permutations with respect to f by $G_{\text{mod}}(f)$. It follows that $G_{\text{mod}}(f)$ is a group for all $f \in \text{Sym}(\mathbb{Z})$; firstly $id_{\mathbb{Z}} \in G_{\text{mod}}(f)$, since $id_{\mathbb{Z}}^f = f id_{\mathbb{Z}} f^{-1} = id_{\mathbb{Z}} \in G_{\text{mod}}$. Secondly, if $g \in G_{\text{mod}}(f)$ then $g^{-1} \in G_{\text{mod}}(f)$, since

$$(g^{-1})^f = fg^{-1}f^{-1} = (fgf^{-1})^{-1} = (g^f)^{-1} \in G_{\text{mod}}$$

Lastly, if $g, h \in G_{\text{mod}}(f)$ then $gh \in G_{\text{mod}}(f)$, since

$$(gh)^f = fghf^{-1} = (fgf^{-1})(fhf^{-1}) = g^f h^f \in G_{\text{mod}}$$

A straightforward calculation shows that $G_{\text{mod}}(f) = fG_{\text{mod}}f^{-1}$, so $G_{\text{mod}} \cong G_{\text{mod}}(f)$ for all $f \in \text{Sym}(\mathbb{Z})$. In other words, any re-ordering of $\mathbb{Z}$ results in a group of 'modular' permutations isomorphic to G_{mod}.

Example 2.3. *Let*

$$f = (-1, 1)(-3, 3)(-5, 5)(-7, 7) \cdots$$

and let

$$g = (\cdots, -4, 3, -2, 1, 0, -1, 2, -3, 4, \cdots)$$

Then g is modular with respect to f, since

$$g^f = (\cdots, -3, -2, -1, 0, 1, 2, 3, \cdots)$$

and g^f is a modular permutation, with modulus 1.

The following result is an expansion of an argument in Fiddes' thesis [8]; the method of proof will be useful when we generalise the concept of a modular permutation in Chapter 4. Notice that if f is a permutation of the integers and c is a cycle involved in f, then $c(x) = f(x)$ for all $x \in \text{supp}(c)$. This proposition is certainly natural, but it is not quite obvious.

Proposition 2.2. *Let $f \in G_{mod}$, suppose that f has modulus n and suppose that c is an infinite cycle involved in f. Then $c \in G_{mod}$, and c has modulus a multiple of n.*

Proof. Let $z_0 \in \operatorname{supp}(c)$. As c is an infinite cycle, by the Dirichlet Principle there exist integers j, k such that $c^j(z_0) \equiv c^k(z_0) \bmod n$. Therefore we can write $c^k(z_0) = c^j(z_0) + rn$ for some $r \in \mathbb{Z}$. Without loss of generality we may assume that $r \in \mathbb{N}$ (otherwise we reverse the labels of j and k). We prove that c has modulus rn, which we do by showing $c(x + rn) = c(x) + rn$ for all $x \in \mathbb{Z}$.

It is sufficient to show that $x \in \operatorname{supp}(c)$ if, and only if, $x + rn \in \operatorname{supp}(c)$. If this statement is true, then for all $x \in \operatorname{supp}(c)$

$$c(x + rn) = f(x + rn) = f(x) + rn = c(x) + rn$$

by Corollary 2.1, and for all $x \notin \operatorname{supp}(c)$

$$c(x + rn) = x + rn = c(x) + rn$$

Let $x \in \operatorname{supp}(c)$, so $x = c^m(z_0)$ for some $m \in \mathbb{Z}$. As c is involved in f we have $c(x) = f(x)$ for all $x \in \mathbb{Z}$, thus

$$x + rn = c^m(z_0) + rn = f^m(z_0) + rn$$

Corollary 2.1 implies that

$$f^m(z_0) + rn = f^m(z_0 + rn) = f^{m-j}(f^j(z_0) + rn)$$

and since both z_0 and $c^j(z_0) + rn$ are elements of $\operatorname{supp}(c)$ we have

$$f^{m-j}(f^j(z_0) + rn) = c^{m-j}(c^j(z_0) + rn) = c^l(c^m(z_0))$$

where $l := k - j$. Therefore $x + rn = c^l(c^m(z_0)) \in \operatorname{supp}(c)$. For the reverse implication, it is sufficient to prove $x \in \operatorname{supp}(c)$ if, and only if, $x - rn \in \operatorname{supp}(c)$. We have

$$x = c^m(z_0) = c^{-l}(c^l(c^m(z_0))) = c^{-l}(c^m(z_0) + rn)$$

Now z_0 and $c^m(z_0) + rn$ are elements of $\mathrm{supp}(c)$, so

$$\begin{aligned} c^{-l}(c^m(z_0) + rn) &= f^{-l}(f^m(z_0) + rn) \\ &= f^{m-l}(z_0 + rn) \\ &= f^{m-l}(z_0) + rn \\ &= c^{m-l}(z_0) + rn \end{aligned}$$

by Corollary 2.1. Thus $x = c^{m-l}(z_0)+rn$, which implies that $x-rn = c^{m-l}(z_0) \in \mathrm{supp}(c)$. Hence $x \in \mathrm{supp}(c)$ if, and only if $x + rn \in \mathrm{supp}(c)$ and the result follows. □

We observed in section 2.1 that we can define an epimorphism $\phi : G_{\mathrm{fin}} \to \mathbb{Z}$, $\phi(f) = \phi_x(f)$ for some $x \in \mathbb{Z} + \frac{1}{2}$. Since every modular permutation has finite flow, one can restrict ϕ to a map $G_{\mathrm{mod}} \to \mathbb{Z}$, and it is easy to see that this is still an epimorphism: for each $z \in \mathbb{Z}$ the permutation $f : \mathbb{Z} \to \mathbb{Z}, f(a) = a + z$ for all $a \in \mathbb{Z}$ is modular with modulus z, and $\phi(f) = z$. We denote the kernel of this map by $G_{\mathrm{mod}}(0)$, and note that $G_{\mathrm{mod}}(0)$ is the group of modular permutations with flow 0. In the next section, we see that the structure of $G_{\mathrm{mod}}(0)$ is very tame.

2.3 Simplicity of $G_{\mathrm{mod}}(0)$

For the sake of notational convenience, for the rest of this section G will denote $G_{\mathrm{mod}}(0)$. In this section we prove that G is a simple group. The first step is to determine a set of generators for G . We show that G is generated by two families of permutations, the first of which are the family of modular extensions, which we define below.

Definition 2.6. *Let $n \in \mathbb{N}$ and suppose that $g \in S_n$. The* **modular extension** *of g is the modular permutation of $\mathbb{Z}$ that sends z to $z + (g(k) - k)$ for all $z \in \mathbb{Z}$, where $z \equiv k \bmod n$. We denote the modular extension of g by $[g]_n$.*

In other words, the modular extension of $g \in S_n$ is a permutation of $\mathbb{Z}$ that 'looks like' g being repeated every n integers, as we can see in the following example.

Example 2.4. *Let $n = 6$ and $g = (123)(56) \in S_6$. Then*

$$[g]_6 = \cdots (-5, -4, -3)(-1, 0)(1, 2, 3)(5, 6)(7, 8, 9)(11, 12) \cdots$$

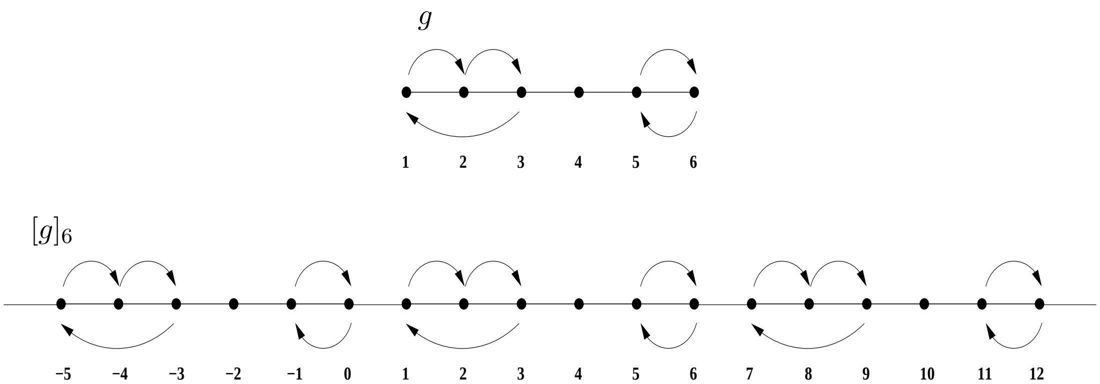

Figure 2-2: Geometric interpretation of g and its modular extension.

It is easy to see in the above example that $[g]_6$ is a modular permutation with modulus 6, and that $[g]_6$ has zero flow. We now see that this observation is true in general.

Lemma 2.7. *Let $n \in \mathbb{N}$ and suppose $g \in S_n$. Then $[g]_n \in G$ and $[g]_n$ has modulus n.*

Proof. Let $z \in \mathbb{Z}$ and suppose that $z + n \equiv k \bmod n$ for some $k \in \{1, 2, \cdots, n\}$. It follows that $z \equiv k \bmod n$ also, and so

$$\begin{aligned} [g]_n(z+n) &= (z+n) + (g(k) - k) \\ &= (z + (g(k) - k)) + n \\ &= [g]_n(z) + n \end{aligned}$$

and so by Definition 2.4 $[g]_n$ is modular with modulus n. Furthermore, it is easy to see from the construction of $[g]_n$ that

$$\phi^+_{n+\frac{1}{2}}([g]_n) = \phi^-_{n+\frac{1}{2}}([g]_n) = 0$$

and so $[g]_n$ has zero flow.Thus $[g]_n \in G$ for all $n \in \mathbb{N}$ and $g \in S_n$. $\square$

The second family of permutations that form part of our generating set for G occur as the kernels of a family of homomorphisms. For each $n \in \mathbb{N}$, denote the set of all the modular permutations of $\mathbb{Z}$ with zero flow that have modulus n by $G^{(n)}$. As might be expected, $G^{(n)}$ forms a group.

Proposition 2.3. *For each $n \in \mathbb{N}$, $G^{(n)}$ is a group.*

Proof. The identity permutation has modulus 1 as observed in the proof of Proposition 2.1, and has zero flow because it fixes every element of $\mathbb{Z}$. If $f, g \in G^{(n)}$ then their composition $g \circ f$ has zero flow by Lemma 2.1, and $g \circ f$ has modulus n, since

$$(g \circ f)(x + n) = g(f(x) + n) = (g \circ f)(x) + n$$

for all $x \in \mathbb{Z}$. If f has modulus n then f^{-1} also has modulus n, as observed in the proof of Proposition 2.1; if f has zero flow then f^{-1} has zero flow, since $\phi(g^{-1}) = -\phi(g)$, and so $G^{(n)}$ is closed under taking inverses. □

Without loss of generality we may consider S_n to be the symmetric group on $\mathbb{Z}_n$, the integers modulo n. For each $n \in \mathbb{N}$ there is a natural map from $G^{(n)}$ into S_n, which we now define.

Definition 2.7. *For each $n \in \mathbb{N}$ we define the map $\varphi_n : G^{(n)} \to S_n$, $g \mapsto \pi_g$, where*

$$\pi_g : \mathbb{Z}_n \to \mathbb{Z}_n \ , \ [z] \mapsto [g(z)]$$

and where $[z]$ is the congruence class of the integer z modulo n.

Example 2.5. *Let $n = 5$, and let*

$$g = (\cdots -4, 3, 1, 8, 6, 13, \cdots) \circ (\cdots 15, 10, 5, 0, -5, \cdots) \circ (\cdots (-3, -1)(2, 4)(7, 9) \cdots)$$

Then g has zero flow and is modular with modulus 5, so $g \in G^{(5)}$. Then $\varphi_5(g)$ is *the permutation π_g of $\mathbb{Z}_5$ that sends*

$$\begin{aligned}
[1] \mapsto& [g(1)] = [8] = [3] \\
[2] \mapsto& [g(2)] = [4] \\
[3] \mapsto& [g(3)] = [1] \\
[4] \mapsto& [g(4)] = [2] \\
[5] \mapsto& [g(5)] = [0] = [5]
\end{aligned}$$

and so, abusing notation slightly, $\pi_g = (13)(24) \in S_5$.

As we can see from the above example, $\varphi_n(g)$ is a permutation of $\{1, 2, \cdots, n\}$ that mimics how g moves the elements of the equivalence classes modulo n. Since a modular permutation of $\mathbb{Z}$ with modulus n is determined solely by how it moves the elements of $\{1, 2, \cdots, n\}$, it makes intuitive sense that each element of G describes an element of S_n. However, we can be more precise as well.

For each $g \in G^{(n)}$ the map π_g is well-defined: if $[z_1] = [z_2]$ for some $z_1, z_2 \in \mathbb{Z}$, then z_1 and z_2 are congruent modulo n. Since g has modulus n, this implies that $g(z_1)$ and $g(z_2)$ are congruent modulo n, hence $[g(z_1)] = [g(z_2)]$ and so $\pi_g([z_1]) = \pi_g([z_2])$. Therefore Definition 2.7 makes sense.

As we might expect, the maps φ_n are not simply maps between groups.

Lemma 2.8. *For each $n \in \mathbb{N}$ the map φ_n is a group homomorphism.*

Proof. Let $f, g \in G^{(n)}$. We wish to show that $\varphi_n(gf) = \varphi_n(g)\varphi_n(f)$, i.e. that $\pi_{gf} = \pi_g \pi_f$. Let $[z] \in \mathbb{Z}_n$. Then

$$\pi_{gf}([z]) = [(gf)(z)] = [g(f(z))] = \pi_g([f(z)]) = \pi_g(\pi_f([z])) = \pi_g\pi_f([z])$$

So $\pi_{gf} = \pi_g \pi_f$ and hence $\varphi_n(gf) = \varphi_n(g)\varphi_n(f)$ □

As a consequence Lemma 2.8 implies that Ker φ_n is a normal subgroup of $G^{(n)}$ for each $n \in \mathbb{N}$. These kernels, along with the modular extensions, form our generating set for G.

Theorem 2.2. *G is generated by the set of all modular extensions, along with the groups Ker φ_n, for each $n \in \mathbb{N}$.*

Proof. Let g be a non-trivial element of G . Suppose that g has modulus n and that $\varphi_n(g) = \sigma$. Now $\varphi_n([\sigma]_n) = \sigma$, and so

$$id_{\mathbb{Z}} = \varphi_n(g)(\varphi_n([\sigma]_n))^{-1} = \varphi_n(g)\varphi_n([\sigma]_n^{-1}) = \varphi_n(g[\sigma]_n^{-1})$$

which implies that $g[\sigma]_n^{-1} \in \text{Ker } \varphi_n$, i.e. $g = [\sigma]_n\tau$ for some $\tau \in \text{Ker } \varphi_n$. □

Recall that if G is a group and $g \in G$ then $\langle g \rangle^G$ denotes the normal closure of g in the group G, the group generated by g and all the conjugates of g by elements of G. To finish the argument that G is simple we show that, for each non-trivial element $g \in G$, $\langle g \rangle^G$ contains every modular extension and the groups Ker φ_n for each $n \in \mathbb{N}$. It then follows from Theorem 2.2 that $\langle g \rangle^G$ contains every element of G. Since any normal subgroup of G contains at least one normal closure, it follows that G is simple.

We now wish to show that for each non-trivial $g \in G$, the group $\langle g \rangle^G$ contains the set of modular extensions. Before we do that we make a remark.

Remark 2.1. *For any $n \in \mathbb{N}$, if two elements of S_n are conjugate, then their modular extensions are also conjugate; more precisely, if $\tau = \sigma^\alpha$ for $\tau, \sigma, \alpha \in S_n$, then $[\tau]_n = [\sigma]_n^{[\alpha]_n}$. In particular, this means that if $\tau, \sigma \in S_n$ have identical cycle structure and $[\tau]_n \in \langle g \rangle^G$ for some non-trivial $g \in G$, then $[\sigma]_n \in \langle g \rangle^G$.*

Proposition 2.4. *Let g be a non-trivial element of G and let $m \in \mathbb{N}$. Then $[f]_m \in \langle g \rangle^G$ for all $f \in S_m$.*

Proof. Notice that Remark 2.1 implies that it is sufficient to show that $[(12)]_m \in \langle g \rangle^G$: if we show this, then $\langle g \rangle^G$ contains all the modular extensions $[\tau]_m$ where τ is a transposition in S_m, and therefore contains every modular extension $[f]_m$ for any $f \in S_m$ because we may write these as a product of modular extensions of transpositions.

Assume that g has modulus n, and take a natural number r such that $rn \geq 4$ and there exists $a \in \{1, 2, \cdots, rn\}$ such that $g(a) \not\equiv a \bmod rn$. Notice that any natural number s such that $r|s$ also has the same property; this will be important later. Let

$$\varphi = \cdots (a-2rn, a-rn)(a, a+rn)(a+2rn, a+3rn)\cdots$$

The permutation φ has zero flow and is modular with modulus $2rn$, hence $\varphi \in G$. Lemma 2.4 tells us that

$$\varphi^g = (\cdots g(a)-2rn, g(a)-rn)(g(a), g(a)+rn)(g(a)+2rn, g(a)+3rn)\cdots$$

Let b be an element of $\{1, 2, \cdots, rn\}$ such that $b \equiv g(a) \bmod rn$. The supports of φ and φ^g are disjoint; if this were not the case, then one of the following conditions would hold:

1. $a \equiv g(a) \bmod 2rn$
2. $a \equiv g(a) + rn \bmod 2rn$
3. $a + rn \equiv g(a) \bmod 2rn$
4. $a + rn \equiv g(a) + rn \bmod 2rn$

It is easy to check for each condition that it holds only if $a \equiv b \bmod rn$, which is false by assumption. Therefore the support of φ and φ^g are disjoint and so

$$\varphi\varphi^g = \cdots (a-2rn, a-rn)(g(a)-2rn, g(a)-rn)(a, a+rn)(g(a), g(a)+rn)\cdots$$

Furthermore $\varphi\varphi^g$ is an element of $\langle g\rangle^G$, because $\varphi\varphi^g = g^\varphi g^{-1}$ and we already observed that $\varphi \in G$.

Since $g(a) \equiv b \bmod rn$, we know that $g(a) = b + trn$ for some integer t. Since $rn \geq 4$ we can choose $c \in \{1, 2, \cdots, rn\}$ such that $c \neq a, b$. Let

$$\sigma = (\cdots, b+rn, b, b-rn, \cdots) \circ (\cdots, c-rn, c, c+rn, \cdots)$$

The permutation σ is a modular permutation with modulus rn and has zero flow, so $\sigma \in G$ and therefore, if we conjugate $\varphi\varphi^g$ by σ^t we have an element of $\langle g\rangle^G$.

A straightforward calculation shows that

$$\begin{aligned}(\varphi\varphi^g)^{\sigma^t} &= \cdots(a-2rn, a-rn)(b-2rn, b-rn)(a, a+rn)(b, b+rn), \cdots \\ &= [(a, a+rn)(b, b+rn)]_{2rn}\end{aligned}$$

and so $[(a, a+rn)(b, b+rn)]_{2rn} \in \langle g\rangle^G$. Remark 2.1 implies that $\langle g\rangle^G$ also contains

$$[(1,2)(1+rn, 2+rn)]_{2rn} = [(12)]_{rn}$$

and so $\langle g\rangle^G$ contains $[f]_{rn}$ for all $f \in S_{rn}$ since each one can be written as a product of modular extensions of transpositions, as we observed earlier.

As we already observed, any natural number s such that $r|s$ has the same property as r, namely that $g(a) \not\equiv a \bmod sn$. It follows that the same result is true when we replace r with s: in particular the same is true when we replace r with mr, i.e. $\langle g\rangle^G$ contains $[f]_{mrn}$ for all $f \in S_{mrn}$. This means that $\langle g\rangle^G$ contains the permutation

$$[(1,2)(1+m, 2+m)\cdots(1+(rn-1)m, 2+(rn-1)m)]_{mrn} = [(12)]_m$$

and so Remark 2.1 implies that $\langle g\rangle^G$ contains $[f]_m$ for all $f \in S_m$. $\square$

Our next aim is to show that for any non-trivial element g of G, $k \in \langle g\rangle^G$ for all $k \in \text{Ker } \varphi_n$ and for all $n \in \mathbb{N}$; in other words

Proposition 2.5. *Let g be a non-trivial element of G and let $n \in \mathbb{N}$. If $f \in$ Ker φ_n, then $f \in \langle g\rangle^G$.*

We prove this in two steps, and we also need a preliminary definition:

Definition 2.8. *Let $n \in \mathbb{N}$, $a, b \in \{1, 2, \cdots, n\}$ such that $a \neq b$. Define the permutation of the integers $\theta_n(a,b)$ as follows:*

$$\theta_n(a,b) := (\cdots, a-n, a, a+n, \cdots) \circ (\cdots b+n, b, b-n, \cdots)$$

It is easy to see that $\theta_n(a,b) \in G$ for each $n \in \mathbb{N}$ and for each $a, b \in \{1, 2, \cdots, n\}$ such that $a \neq b$: By inspection we can see that $\theta_n(a,b)$ has (principal) modulus n. Furthermore we can see that the only integer that $\theta_n(a,b)$

moves past $\frac{1}{2}$ in the positive direction is $a-n$, and the only integer that $\theta_n(a,b)$ moves past $\frac{1}{2}$ in the negative direction is b. Hence $\phi_{\frac{1}{2}}(\theta_n(a,b)) = 1-1 = 0$ and so $\theta_n(a,b)$ has zero flow by Lemma 2.3 and Lemma 2.2.

The first of the two steps in proving Proposition 2.5 is to prove that, for a non-trivial element g of G, that each $\theta_n(a,b)$ is contained in $\langle g\rangle^G$.

Lemma 2.9. *Let g be a non-trivial element of G. Then $\theta_n(a,b) \in \langle g\rangle^G$ for all $n \in \mathbb{N}$ and for all $a, b \in \{1, 2, \cdots, n\}$ such that $a \neq b$.*

Proof. Let g be a non-trivial element of G and let $n \in \mathbb{N}$. Firstly, we show that if $\theta_n(a,b) \in \langle g\rangle^G$ for some distinct $a, b \in \{1, 2, \cdots, n\}$, then $\theta_m(x,y) \in \langle g\rangle^G$ for all distinct $x, y \in \{1, 2, \cdots, n\}$. Let x, y be as above. Without loss of generality we can assume that $\{x,y\} \neq \{a,b\}$: If these two sets are equal, then either $x = a$ and $y = b$, or $x = b$ and $y = a$. In the first case $\theta_n(x,y) = \theta_n(a,b)$, and in the second case $\theta_n(x,y) = (\theta_n(a,b))^{-1}$. The two possibilities if $\{x,y\} \neq \{a,b\}$ are that $\{x,y\}$ and $\{a,b\}$ are disjoint, or that $\{x,y\} \cap \{a,b\}$ contains one element.

If $\{x,y\}$ and $\{a,b\}$ are disjoint, then

$$\theta_n(x,y) = \theta_n(a,b)^\tau$$

where

$$\tau = [(a,x)(b,y)]_n$$

The map τ is a modular extension. We observed before that the set of modular extensions is contained in G, and so $\theta_n(x,y) \in \langle g\rangle^G$.

If $\{x,y\} \cap \{a,b\}$ contains one element, then we can assume without loss of generality that $x = a$. For, if $x = b$, then we can switch the labels of a and b and work with $(\theta_n(a,b))^{-1}$ instead. If $y = a$, then we can switch the labels for x and y and work with $(\theta_n(x,y))^{-1}$ instead. Finally if $y = b$, then we can switch both sets of labels and work with the inverses of both permutations instead. Thus, if $x = a$, then

$$\theta_n(a,y) = \theta_n(a,b)^\sigma$$

where

$$\sigma = [(b, y)]_n$$

Similarly to the previous case, the map σ is a modular extension. All of the modular extensions are contained in G and so $\theta_n(x, y) \in \langle g \rangle^G$.

The above argument implies that, for each $n \in \mathbb{N}$, we only need to show that one element of the form $\theta_n(a, b)$ is contained in $\langle g \rangle^G$. We divide this into four separate cases: $n = 2$, $n = 3$, $n = 4$ and $n \geq 5$.

When $n = 2$, let

$$h_1 = \cdots (0,1)(2,3)(8,9)(10,11) \cdots$$

$$h_2 = [(4,5)(6,7)]_8$$

$$h_3 = [(1,2)]_2$$

We show that h_1, h_2 and h_3 are elements of $\langle g \rangle^G$, and that $h_1 \circ h_2 \circ h_3 = \theta_2(1,2)$. The maps h_2 and h_3 are modular extensions, and so are elements of $\langle g \rangle^G$ by Proposition 2.4. We can write $h_1 = \alpha^\beta$, where

$$\alpha = [(1,2)(3,4)]_8$$

$$\beta = \cdots (12,11,10,9,8)(4,3,2,1,0) \cdots$$

The map α is a modular extension, and so is an element of $\langle g \rangle^G$ by Proposition 2.4. By inspection we see that β has zero flow and is modular with modulus 8, so $\beta \in G$. Therefore $h_1 \in \langle g \rangle^G$, and so $h_1 \circ h_2 \circ h_3 \in \langle g \rangle^G$. A straightforward calculation shows that

$$\begin{aligned}
h_1 \circ h_2 \circ h_3 &= (\cdots (0,1)(2,3)(8,9)(10,11) \cdots) \circ [(4,5)(6,7)]_8 \circ [(1,2)]_2 \\
&= (\cdots (0,1)(2,3)(4,5)(6,7) \cdots) \circ [(1,2)]_2 \\
&= (\cdots, -3, -1, 1, 3, \cdots) \circ (\cdots, 4, 2, 0, -2, -4, \cdots) \\
&= \theta_2(1,2)
\end{aligned}$$

and so we are done.

When $n = 3$, let

$$h_1 = \cdots (8,10)(11,13)(20,22)(23,25)\cdots$$

$$h_2 = [(2,4)(5,7)]_{12}$$

$$h_3 = [(1,2)]_3$$

Similarly to the $n = 2$ case, we show that h_1, h_2 and h_3 are elements of $\langle g\rangle^G$, and that $h_1 \circ h_2 \circ h_3 = \theta_3(1,2)$. Again, the maps h_2 and h_3 are modular extensions, and so are elements of $\langle g\rangle^G$ by Proposition 2.4. We can write $h_1 = h_2^\alpha$, where

$$\alpha = \cdots (2,8)(4,10)(5,11)(7,13)(14,20)(16,22)(17,23)(19,25)\cdots$$

By inspection we see that α has zero flow, and is modular with modulus 12, so $\alpha \in G$. Therefore $h_1 \in \langle g\rangle^G$, and so $h_1 \circ h_2 \circ h_3 \in \langle g\rangle^G$. A straightforward calculation shows that

$$\begin{aligned} h_1 \circ h_2 \circ h_3 &= (\cdots (8,10)(11,13)(20,22)(23,25)\cdots) \circ [(2,4)(5,7)]_{12} \circ [(1,2)]_3 \\ &= (\cdots (2,4)(5,7)(8,10)(11,13)\cdots) \circ [(1,2)]_3 \\ &= (\cdots, -5, -2, 1, 4, 7, \cdots) \circ (\cdots 8, 5, 2, -1, -4, \cdots) \\ &= \theta_3(1,2) \end{aligned}$$

and so we are done.

When $n = 4$, let

$$h_1 = \cdots (10,13)(14,17)(26,29)(30,33)\cdots$$

$$h_2 = [(2,5)(6,9)]_{16}$$

$$h_3 = [(1,2)]_4$$

Once again, we show that h_1, h_2 and h_3 are elements of $\langle g\rangle^G$, and then show that $h_1 \circ h_2 \circ h_3 = \theta_4(1,2)$. As before, the maps h_2 and h_3 are modular extensions, and so are elements of $\langle g \rangle^G$ by Proposition 2.4. We can write $h_1 = h_2^\alpha$, where

$$\alpha = \cdots (2,10)(5,13)(6,14)(9,17)(18,26)(21,29)(22,30)(25,33) \cdots$$

By inspection we see that α has zero flow and is modular with modulus 16, so $\alpha \in G$. Therefore $h_1 \in \langle g \rangle^G$, and so $h_1 \circ h_2 \circ h_3 \in \langle g \rangle^G$. A straightforward calculation show that

$$\begin{aligned} h_1 \circ h_2 \circ h_3 &= (\cdots (10,13)(14,17)(26,29)(30,33) \cdots) \circ [(2,5)(6,9)]_{16} \circ [(1,2)]_4 \\ &= (\cdots (2,5)(6,9)(10,13)(14,17) \cdots) \circ [(1,2)]_4 \\ &= (\cdots, -7, -3, 1, 5, 9, \cdots) \circ (\cdots, 10, 6, 2, -2, -6, \cdots) = \theta_4(1,2) \end{aligned}$$

and so we are done.

When $n \geq 5$, let

$$h_1 = [(1,2)(3,4)]_n$$

$$h_2 = \cdots (1, 2+n)(3,4)(1+n, 2+2n)(3+n, 4+n) \cdots$$

We show that h_1 and h_2 are elements of $\langle g \rangle^G$ and that $h_1 \circ h_2 = \theta_n(1,2)$. The map h_1 is a modular extension, and so is an element of $\langle g \rangle^G$ by Proposition 2.4. We can write $h_2 = h_1^\alpha$, where

$$\alpha = \theta_n(2,5)$$

We observed before that $\theta_n(a,b) \in G$ for all distinct $a, b \in \{1, 2, \cdots, n\}$, and so $h_2 \in \langle g \rangle^G$. Therefore $h_1 \circ h_2 \in \langle g \rangle^G$, and a straightforward calculation shows that

$$\begin{aligned} h_1 \circ h_2 &= [(1,2)(3,4)]_n \circ (\cdots (1, 2+n)(3,4)(1+n, 2+2n)(3+n, 4+n) \cdots) \\ &= (\cdots, 1-n, 1, 1+n, \cdots) \circ (\cdots, 2+n, 2, 2-n, \cdots) \\ &= \theta_n(1,2) \end{aligned}$$

and so we are done. Hence for each $n \in \mathbb{N}$, the map $\theta_n(1,2) \in \langle g \rangle^G$, and so it follows from the argument at the beginning of the proof that $\theta_n(a,b) \in \langle g \rangle^G$ for all distinct $a, b \in \{1, 2, \cdots, n\}$. $\square$

The next preliminary step in proving Proposition 2.5 is to determine a normal form for the elements of Ker φ_n, for each $n \in \mathbb{N}$.

Lemma 2.10. *Let $n \in \mathbb{N}$ and $f \in G^{(n)}$. Then $f \in$ Ker φ_n if and only if f can be written in the form*

$$f = (\cdots 1-n, 1, 1+n, \cdots)^{x_1} \circ (\cdots 2-n, 2, 2+n, \cdots)^{x_2} \circ \cdots \circ (\cdots 0, n, 2n, \cdots)^{x_n}$$

for some $i \in \{1, 2, \cdots, n\}$, where $\sum_{i=1}^{n} x_i = 0$.

Proof. We begin by proving the necessary condition. Let $f \in \text{Ker } \varphi_n$, let $i \in \{1, 2, \cdots, n\}$ and suppose that f moves i. As $\varphi_n(f) = id_n$, it follows that $f(i) = i + x_i n$ for some $x_i \in \mathbb{Z}$. Also, f is modular with modulus n, and so

$$f(i + zn) = f(i) + zn = i + (x_i + z)n$$

for all $z \in \mathbb{Z}$. Therefore, the cycles

$$(\cdots, i + (s - x_i)n, i + sn, i + (s + x_i)n, \cdots)$$

are involved in f, for each $s \in \{1, 2, \cdots, n\}$. Since i was an arbitrarily chosen element of $\{1, 2, \cdots, n\}$, it follows that

$$f = (\cdots 1-n, 1, 1+n, \cdots)^{x_1} \circ (\cdots 2-n, 2, 2+n, \cdots)^{x_2} \circ \cdots \circ (\cdots 0, n, 2n, \cdots)^{x_n}$$

Now f has zero flow, but each factor $(\cdots i-n, i, i+n, \cdots)^{x_i}$ has flow x_i. We know that composing finite flow permutations sums their flow, so it must be that $\sum_{i=1}^{n} x_i = 0$.

Now we prove the sufficient condition. Let f be a permutation of the form in the statement of the lemma, and suppose that $\sum_{i=1}^{n} x_i = 0$. By inspection, we see that f is modular with modulus n. Each factor $(\cdots i-n, i, i+n, \cdots)^{x_i}$ is a permutation with flow x_i, and as $\sum_{i=1}^{n} x_i = 0$ it follows that f has zero flow. Therefore $f \in G^{(n)}$. Since every element moved by f is moved by distance a multiple of n, it follows that $\varphi_n(f) = id_n$, and so $f \in \text{Ker } \varphi_n$. $\square$

Notice that if f is a permutation of the form in the statement of Lemma 2.10, then $\sum_{i=1}^{n} |x_i|$ is even. We will sometimes refer to the sum $\sum_{i=1}^{n} x_i$ as the **cycle exponent sum** of f. We now have enough tools to prove Proposition 2.5.

Proof of Proposition 2.5. As in the statement of the proposition, let g be a non-trivial element of G and let $f \in \text{Ker } \varphi_n$ for some $n \in \mathbb{N}$. We prove that we can write f as a product of elements of the form $\theta_n(a, b)$. Once we show this, Lemma 2.9 will imply that f is a product of elements of $\langle g \rangle^G$, which will prove the proposition.

Let $n \in \mathbb{N}$. We prove, by induction on k, that if $f \in \text{Ker } \varphi_n$ and $\sum_{i=1}^{n} |x_i| = 2k$, then f is a product of elements of the form $\theta_n(a, b)$.

When $k = 1$, the total number of cycles involved in f is 2. Since $\sum_{i=1}^{n} x_i = 0$, it follows that there are distinct $i, j \in \{1, 2, \cdots, n\}$ such that $\{x_i, x_j\} = \{1, -1\}$. This implies that either

$$f = (\cdots, i-n, i, i+n, \cdots) \circ (\cdots, j+n, j, j-n, \cdots) = \theta_n(i, j)$$

or

$$f = (\cdots, j-n, j, j+n, \cdots) \circ (\cdots, i+n, i, i-n, \cdots) = \theta_n(j, i)$$

so the statemtent is true for $k = 1$.

Now assume that the statement is true for all natural numbers less than k, and suppose that the total number of cycles involved in f is $2k$. Since $\sum_{i=1}^{n} x_i = 0$, it follows that there are distinct $i, j \in \{1, 2, \cdots, n\}$ such that $x_i > 0$ and $x_j < 0$. We can write

$$(\cdots, j-n, j, j+n, \cdots)^{x_j} = (\cdots, j-n, j, j+n, \cdots)^{x_i} \circ (\cdots, j-n, j, j+n, \cdots)^{x_i + x_j}$$

and so

$$\begin{aligned} f &= ((\cdots, i-n, i, i+n, \cdots) \circ (\cdots, j+n, j, j-n, \cdots))^{x_i} \circ g \\ &= (\theta_n(i, j))^{x_i} \circ g \end{aligned}$$

where g is a product of cycles of the form $(\cdots i-n, i, i+n \cdots)$ such that the cycle exponent sum is zero. Lemma 2.10 implies that $g \in \text{Ker } \varphi_n$. As the total number of cycles involved in g is less than $2k$, the induction hypothesis implies that g is a product of elements of the form $\theta_n(a, b)$. Therefore f is a product of elements of the form $\theta_n(a, b)$, and so the statement is proved by induction. $\square$

At last, we are in a position to prove the main theorem of this section.

Theorem 2.3. *G is simple.*

Proof. Let g be a non-trivial element of G. We show that $\langle g \rangle^G$ contains every element of G. Let h be another non-trivial element of G,suppose that h has modulus n and $\varphi_n(f) = \sigma$. Theorem 2.2 tells us that we can write

$$h = [\sigma]_n \circ f$$

for some $f \in \text{Ker } \varphi_n$. Proposition 2.4 tells us that $[\sigma]_n \in \langle g \rangle^G$. Proposition 2.5 implies that f can be written as a product of elements of the form $\theta_n(a, b)$, each of which are elements of $\langle g \rangle^G$ by Lemma 2.9. Therefore $f \in \langle g \rangle^G$, and so $h \in \langle g \rangle^G$. Therefore $\langle g \rangle^G = G$ and since g is arbitrary, G is simple. $\square$

Chapter 3

The cyclizer series of a group acting on an n-branched star

Consider the 'number line' representation of $\mathbb{Z}$: a 2-regular infinite graph with vertex set $\mathbb{Z}$ and edge set as shown in Figure 3-1. It follows that certain results concerning the integers can be interpreted as results about this graph. This motivates the introduction of a family of infinite graphs that generalise the representation of $\mathbb{Z}$ in Figure 3-1.

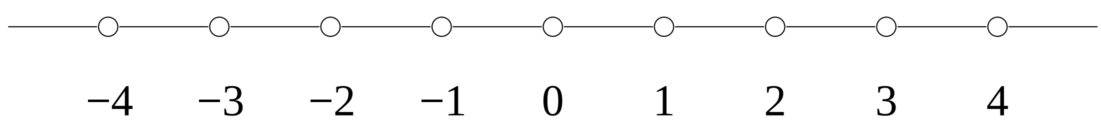

Figure 3-1: A 2-regular infinite graph, with vertex set $\mathbb{Z}$ in the natural way.

3.1 Building an n-branched star

Definition 3.1. *A* **ray** *is an infinite, connected graph with one vertex having valency* 1 *and the remaining vertices having valency* 2*. We call the vertex with valency* 1 *the* **initial vertex** *of the ray.*

If we have n disjoint rays and an additional vertex O, we can build a new graph Γ_n by joining the initial vertex of each ray to O via an edge. This motivates the following definition:

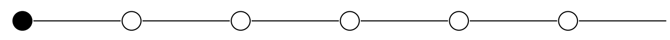

Figure 3-2: Initial fragment of a ray, with initial vertex shaded

Definition 3.2. *Let X be a graph and $n \in \mathbb{N}$. We say that X is an n-***branched star** *if it is graph isomorphic to Γ_n.*

Notice that if $n \geq 3$ then an n-branched star has a unique vertex with valency n (all the other vertices have valency 2). We call this vertex the **origin** of an n-branched star.

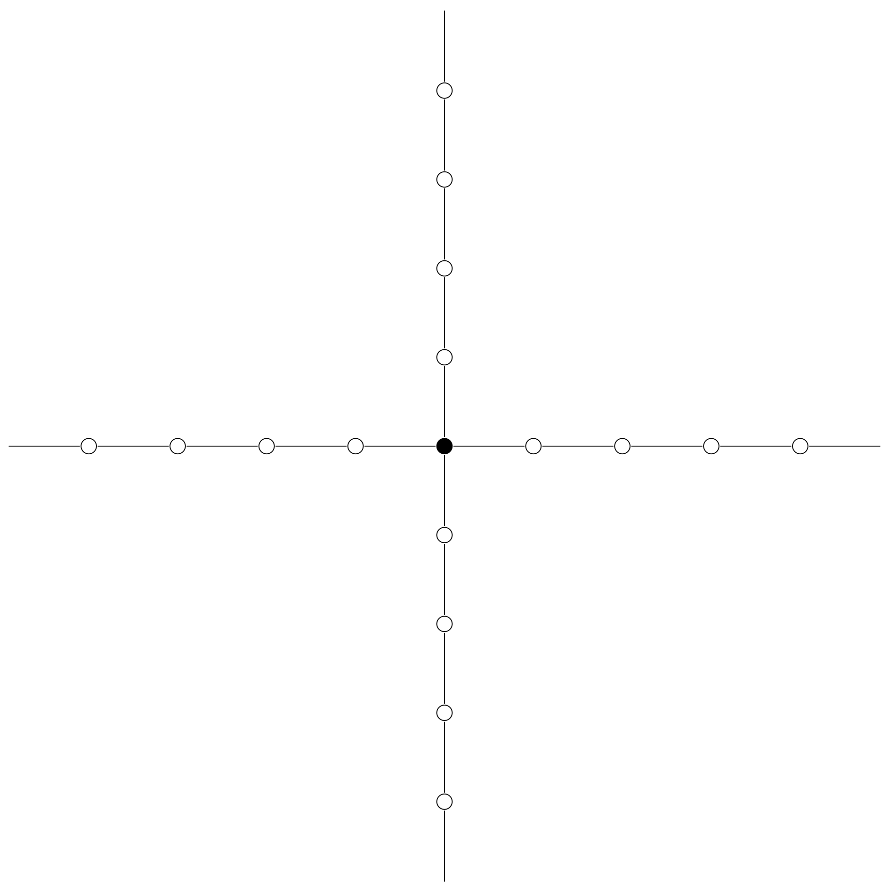

Figure 3-3: Central fragment of a 4-branched star, with origin shaded

Definition 3.3. *Let X be an n-branched star for some $n \geq 3$. A* **branch** X_i **of** *X is a subgraph that does not contain the origin of X, is graph isomorphic to a ray and whose initial vertex is adjacent (in X) to the origin of X.*

In other words, a branch of an n-branched star is one of the rays that is joined to the origin. We will often denote the vertex set of an n-branched star by V_n. Notice that an n-branched star has n disjoint branches, and so V_n is the union of the vertex sets of all the branches, along with the origin. Notice also that a 2-branched star is graph isomorphic to the 'number line' graph in Figure 3-1.

For much of this chapter it will be necessary to talk about specific vertices of an n-branched star, and so we will need a sensible vertex labelling. We do this in the following way: First, label the branches arbitrarily as $X_1, X_2, \cdots, X_n$. Label the origin with the label O and label any non-origin v with the label $v_{r,x}$, where X_r is the branch containing v and x is the length of the shortest path from v to O. For further notational convenience we will sometimes denote the origin O by

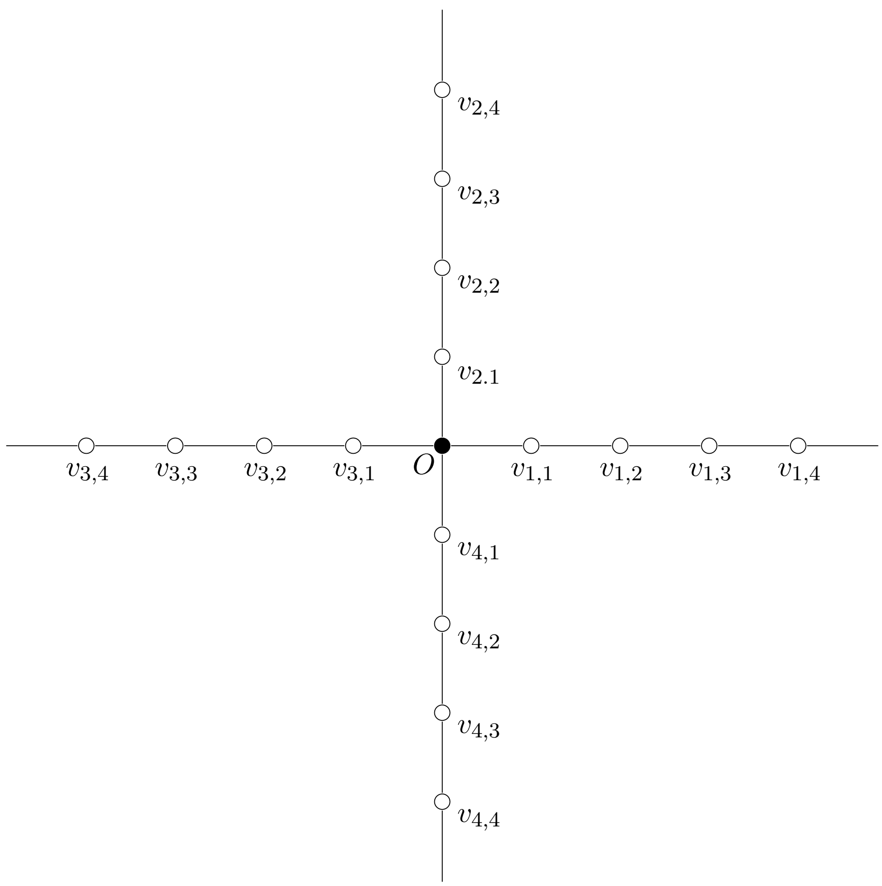

Figure 3-4: Fragment of a 4-branched star with one possible labelling

$v_{r,o}$, for some $r \in \{1, 2, \cdots, n\}$. When $n = 2$, it will often be more convenient to use the integer labelling for the 2-branched star as shown in Figure 3-1, and so we will be freely alternating between using both labellings as we see fit.

3.2 A permutation group of the vertex set of an n-branched star

For the remainder of this chapter let Γ_n denote a labelled n-branched star, let $X_1, X_2, \cdots X_n$ denote the branches of Γ_n and let V_n denote the vertex set of Γ_n. We define some interesting permutations of V_n.

Definition 3.4. *Let r, s be distinct elements of $\{1, 2, \cdots, n\}$. We define $\sigma_{r,s} : V_n \to V_n$,*

$$\sigma_{r,s} = (\cdots, v_{r,2}, v_{r,1}, O, v_{s,1}, v_{s,2}, \cdots)$$

When $s \equiv r + 1 \pmod n$ we write σ_r instead.

Now we define the permutation group that we shall be investigating for the remainder of this chapter.

Definition 3.5. *Define the group G_n to be the permutation group on V_n generated by the permutations σ_r for all $r \in \{1, 2, \cdots, n\}$. In other words,*

$$G_n := \langle \sigma_1, \sigma_2, \cdots, \sigma_n \rangle$$

Notice that the group G_2 is isomorphic to the permutation group induced by the infinite cyclic group C_∞ acting on itself on the natural way, and hence the cyclizer series of G_2 has already been determined. We will use G_2 and C_∞ interchangeably throughout the rest of the thesis.

Remark 3.1. *For all distinct $r, s \in \{1, 2, \cdots, n\}$ we have $\sigma_{r,s} \in G_n$, since*

$$\sigma_{r,s} = \begin{cases} \sigma_r \circ \sigma_{r+1} \circ \cdots \circ \sigma_{s-1} & \text{if } r < s \\ \sigma_s \circ \sigma_{s+1} \circ \cdots \circ \sigma_{r-1} & \text{if } s < r \end{cases}$$

Next, we charactise the group G_n. Recall that $FS(V_n)$ denotes the finitary symmetric group on V_n, i.e. the group of permutations of V_n that have finite support.

Lemma 3.1. *For $n \geq 3, FS(V_n) \trianglelefteq G_n$.*

Proof. Firstly, for all $s \in \{1, 2, \cdots, n\}$ define $\theta_s := (O, v_{s,1})$. A routine calculation shows that $\theta_s = [\sigma_s, \sigma_{s+1}]$, and so $\theta_s \in G_n$. We claim that

$$\{(O, v_{s,x}) \mid s \in \{1, 2, \cdots, n\} \ , \ x \in \mathbb{N}\} \subseteq G_n$$

To prove this, we fix $s \in \{1, 2, \cdots, n\}$ and prove the proposition

$$P(k) : (O, v_{s,k}) \in G_n \ , \ \forall k \in \mathbb{N}$$

by induction on k. $P(1)$ says $(O, v_{s,1}) \in G_n$, which we have shown to be true above. Now assume $P(k)$ is true for some $k \in \mathbb{N}$ and consider $P(k+1)$:

$$(O, v_{s,k+1}) = (O, v_{s,k})^{(v_{s,k}, v_{s,k+1})}$$

and $(v_{s,k}, v_{s,k+1}) = (O, v_{s,1})^{\sigma_s^k}$. Hence the claim is proved by induction. Now consider the transposition $(v_{s,x}, v_{t,y})$ where $s, t \in \{1, 2, \cdots, n\}$, $x, y \in \mathbb{N}$. Then

$$(v_{s,x}, v_{t,y}) = (O, v_{t,y})^{(O, v_{s,x})}$$

which is an element of G_n by the above claim. This proves that the set

$$H := \{(v_{s,x}, v_{t,y}) \mid s, t \in \{1, 2, \cdots, n\} \ , \ x, y \in \mathbb{N} \cup \{O\}\}$$

is contained in G_n. The set H generates $FS(V_n)$, and so $FS(V_n) \leq G_n$. The normality of $FS(V_n)$ in G_n follows from the fact that conjugation preserves cycle structure, so any conjugate in $\mathrm{Sym}(V_n)$ of a finitary permutation must also be finitary. $\square$

Lemma 3.1 tells us that the derived subgroup G_n' is a finitary group, as we see below.

Corollary 3.1. $G_n' \leq FS(V_n)$

Proof. Let s, t be distinct elements of $\{1, 2, \cdots, n\}$. We show that $[\sigma_s, \sigma_t] \in FS(V_n)$. We may assume that $t \not\equiv s + 1 \pmod{n}$, since this case was dealt with in the proof of Lemma 3.1. Furthermore, since $[\sigma_t, \sigma_s] = [\sigma_s, \sigma_t]^{-1}$ we can assume without loss of generality that $s \not\equiv t+1 \pmod{n}$. Hence $\mathrm{supp}(\sigma_s) \cap \mathrm{supp}(\sigma_t) = \{O\}$, and so

$$\begin{aligned}
[\sigma_s, \sigma_t] &= \sigma_s \sigma_s^{-\sigma_t} \\
&= \sigma_s \circ (\cdots, v_{s+1,2}, v_{s+1,1}, v_{t+1,1}, v_{s,1}, v_{s,2}, \cdots) \\
&= (v_{s+1,1}, v_{t+1,1}, O) \\
&\in FS(V_n)
\end{aligned}$$

□

Armed with this understanding of the derived subgroup G_n', we can determine a normal form for elements of G_n.

Lemma 3.2. *Let $g \in G_n$. Then g can be written in the form*

$$\sigma_1^{s_1} \sigma_2^{s_2} \cdots \sigma_{n-1}^{s_{n-1}} \varepsilon$$

where $s_i \in \mathbb{Z}\ \forall i \in \{1, 2, \cdots, n-1\}$, $\varepsilon \in FS(V_n)$.

Proof. Firstly, observe that $\sigma_1 \sigma_2 \cdots \sigma_n = id_{\mathbb{Z}}$ and thus we can write

$$\sigma_n = \sigma_{n-1}^{-1} \circ \sigma_{n-2}^{-1} \circ \cdots \circ \sigma_1^{-1}$$

and so $G_n = \langle \sigma_1, \sigma_2, \cdots, \sigma_{n-1} \rangle$. Let $g \in G_n$ and consider the coset $gFS(V_n)$. Now $gFS(V_n) \in G_n/FS(V_n)$ and since Corollary 3.1 says $G_n' \leq FS(V_n)$, it follows that $G_n/FS(V_n)$ is abelian. The group $G_n/FS(V_n)$ is generated by the set

$$\{\widehat{\sigma_i} := \sigma_i FS_n \mid i \in \{1, 2, \cdots, n-1\}\}$$

and since $G_n/FS(V_n)$ is abelian it must be the case that

$$gFS(V_n) = \widehat{\sigma_1}^{s_1} \widehat{\sigma_2}^{s_2} \cdots \widehat{\sigma_{n-1}}^{s_{n-1}}$$

for some $s_1, s_2, \cdots, s_{n-1} \in \mathbb{Z}$. Thus

$$gFS(V_n) = \sigma_1^{s_1} \sigma_2^{s_2} \cdots \sigma_{n-1}^{s_{n-1}} FS(V_n)$$

and so

$$g = g.id_{\mathbb{Z}} = \sigma_1^{s_1}\sigma_2^{s_2}\cdots\sigma_{n-1}^{s_{n-1}}\varepsilon$$

for some $\varepsilon \in FS(V_n)$. $\square$

Observe that each of the generators of G_n moves elements of V_n a distance of at most 1 (here, the distance between two vertices of V_n is understood to be the length of the shortest path between them). It follows that any permutation $g \in G_n$ can move elements of V_n by a bounded amount; in particular if $g = \sigma_1^{s_1}\sigma_2^{s_2}\cdots\sigma_{n-1}^{s_{n-1}}\varepsilon$ for $s_1, s_2, \cdots, s_{n-1} \in \mathbb{Z}$ and $\varepsilon \in FS(V_n)$ then g can move an element of V_n a distance at most

$$\sum_{i=1}^{n-1} |s_i| + M_\varepsilon$$

where M_ε is the maximum distance ε moves an element of V_n (note that M_ε is finite because ε is a finitary permutation of V_n). The class of bounded permutations defined below turn out to be important in understanding the cycle structure of elements of G_n.

Definition 3.6. *Let f be a bounded permutation of V_n. We say that f is* **ultimately elementary** *if for all $r \in \{1, 2, \cdots, n\}$ there exists $a_r \in \mathbb{Z}$ such that*

$$f(v_{r,x}) = v_{r,x+a_r}$$

for all $x > N$, where N is some natural number independent of r. We say that N is the **radius of non-uniformity** *for f, and for each $r \in \{1, 2, \cdots, n\}$ we say that a_r is the* **branch speed** *of f on branch X_r.*

In other words, f moves vertices on the branch X_r by translation by a_r, as long as we are sufficiently far away from the origin.

Example 3.1. *Let $n = 3$. Then*

$$g = (\cdots, v_{1,3}, v_{1,2}, v_{1,1}, v_{2,1}, v_{2,3}, v_{2,5}, \cdots) \circ (\cdots, v_{3,3}, v_{3,2}, v_{3,1}, O, v_{2,2}, v_{2,4}, v_{2,6}, \cdots)$$

is ultimately elementary, with $a_1 = -1 = a_3$ and $a_2 = 2$.

Notice that g from Example 3.1 is equal to $\sigma_2^{-1}\sigma_1 \in G_3$. In fact, it will turn out that every element of G_n is an utimately elementary permutation. When dealing with an ultimately elementary permutation f, it sometimes will be useful to partition the vertex set V_n into two subsets, one of which contains only elements that f translates at the respective branch speeds.

Remark 3.2. *Since all the radii of uniformity for f are natural numbers, there is a minimal radius of uniformity; call this N_0. Define the set*

$$\Omega = \{v_{r,x} \in V_n | \ x \leq N_0\}$$

In other words, we describe the smallest ball around the origin of Γ_n, such that every vertex outside this ball is translated by f at the respective branch speeds, and then let Ω be the vertices of Γ_n inside this ball. We call Ω the **ball of non-uniformity***.*

Let E_n be the set of ultimately elementary permutations on X. We can define a function $\phi : E_n \to \mathbb{Z}^n$, $f \mapsto (a_1, a_2, \cdots, a_n)$, where the quantities a_i are defined as in Definition 3.6. Referring back to Example 3.1 once more, observe that for the permutation g we have $a_1 + a_2 + a_3 = 0$. It turns out that a similar equality holds for each element of G_n, which motivates the next definition.

Definition 3.7. *We say that an ultimately elementary permutation f* **satisfies the Kirchoff property** *if $\sum_{k=1}^{n} a_k = 0$, where $a_1, a_2, \cdots, a_n$ are as defined in Definition 3.6. We denote the set of all ultimately elementary permutations that satsify the Kirchoff property by K_n.*

The terminology in Definition 3.7 comes from Kirchoff's First Law, a physical law concerning the conservation of charge in electrical circuits. It says that at any node in an electrical circuit, the sum of currents flowing into that node is equal to the sum of currents flowing out of that node. If we imagine that the branch speeds of an ultimately elementary permutation f represent the size and direction of currents flowing into a node placed at the origin of Γ_n, then satisfying the Kirchoff property means that f is a sensible representation that mimics the physical behaviour of the associated circuit.

Observe that K_n is a group; firstly, $id \in K_n$ since id is an ultimately elementary permutation with $a_1 = a_2 = \cdots = a_n = 0$. Secondly K_n is closed under taking inverses since if f is an ultimately elementary permutation which moves almost all the vertices of branch X_i by a_i, then f^{-1} moves almost all the vertices of branch X_i by $-a_i$. Finally, K_n is closed under composition since if $f, g \in K_n$ and f and g move almost all the vertices of branch X_i by a_i and b_i respectively, then $g \circ f$ moves almost all the vertices of branch X_i by $a_i + b_i$.

Proposition 3.1. $G_n = K_n$

Proof. Firstly, let $f \in G_n$. Using Lemma 3.2, we can write $f = \sigma_1^{s_1}\sigma_2^{s_2}\cdots\sigma_{n-1}^{s_{n-1}}\varepsilon$ where $s_i \in \mathbb{Z}\ \forall i \in \{1, 2, \cdots, n-1\}$ and $\varepsilon \in FS(V_n)$. Let $r \in \{1, 2, \cdots, n\}$.

If $r \neq 1, n$ then there exists $N_r \in \mathbb{N}$ such that for all $k > N_r$,

$$f(v_{r,k}) = \sigma_{r-1}^{s_{r-1}}\sigma_r^{s_r}(v_{r,k})$$

and so there exists $N_r' \in \mathbb{N}$ such that $\forall k > N_r'$

$$f(v_{r,k}) = v_{r,k+s_{r-1}-s_r}$$

Hence for all $k > N_r'$ the directed distance from $v_{r,k}$ to $f(v_{r,k})$ is

$$(k + s_{r-1} - s_r) - k = s_{r-1} - s_r =: a_r$$

If $r = 1$, there exists $N_1 \in \mathbb{N}$ such that for all $k > N_1$,

$$f(v_{1,k}) = \sigma_1^{s_1}(v_{1,k})$$

and so there exists $N_1' \in \mathbb{N}$ such that for all $k > N_1'$,

$$f(v_{1,k}) = v_{1,k-s_1}$$

so for all $k > N_1'$ the directed distance from $v_{1,k}$ to $f(v_{1,k})$ is

$$(n - s_1) - n = -s_1 =: a_1$$

If $r = n$, there exists $N_n \in \mathbb{N}$ such that for all $k > N_n$,

$$f(v_{n,k}) = \sigma_{n-1}^{s_{n-1}}(v_{n,k}) = v_{n,k+s_{n-1}}$$

and so for all $k > N_n$ the directed distance from $v_{n,k}$ to $f(v_{n,k})$ is

$$(n + s_{n-1}) - n = s_{n-1} =: a_n$$

Hence f is ultimately elementary with radius of uniformity $N := \max\{N_1, N_2, \cdots, N_n\}$ and branch speeds $a_1, a_2, \cdots, a_n$ as defined above. Furthermore,

$$\sum_{k=1}^{n} a_k = -s_1 + \sum_{k=2}^{n-1}(s_{k-1} - s_k) + s_{n-1} = -s_1 + (s_1 - s_{n-1}) + s_{n-1} = 0$$

and so f satisfies the Kirchoff property, in other words $f \in K_n$.

Conversely, let $f \in K_n$, so f is ultimately elementary and $\sum_{k=1}^{n} a_k = 0$, where $a_1, a_2, \cdots, a_n$ are the respective branch speeds. Let

$$g = \sigma_1^{-a_1}\sigma_2^{-(a_1+a_2)} \cdots \sigma_{n-1}^{-(\sum_{i=1}^{n-1} a_i)} \in G_n$$

Let N_0 be the minimal radius of non-uniformity for f. Notice that for $k > M_1 = \max\{N_0, a_1\}$, $g(v_{1,k}) = v_{1,k+a_1} = f(v_{1,k})$. Furthermore, notice that for $k > N_0$, $g(v_{n,k}) = v_{n,k+a_n} = f(v_{n,k})$. Finally, notice that for $r \in \{2, 3, \cdots, n-1\}$ and for $k > M_r = \max\{N_0, a_r\}$,

$$g(v_{r,k}) = v_{r,k+(\sum_{k=1}^{r} a_k - \sum_{k=1}^{r-1} a_k)} = v_{r,k+a_r} = f(v_{r,k})$$

and so f acts differently to g on at most a finite number of vertices, i.e. $f = g\varepsilon$ for some $\varepsilon \in FS(V_n)$. By Lemma 3.1, $f \in G_n$. $\square$

Recall Example 3.1; we had an ultimately elementary permutation g of G_3 consisting of two infinite cycles. Consider the cycle involved in g,

$$c := (\cdots, v_{1,3}, v_{1,2}, v_{1,1}, v_{2,1}, v_{2,3}, v_{2,5}, \cdots)$$

Now c is not an ultimately elementary permutation on G_3, since whilst c moves all vertices of branch X_2 of the form $v_{2,2k}$ for $k \in \mathbb{N}$, it does not move vertices of X_2 of the form $v_{2,2k+1}$, for $k \in \mathbb{N}$. It follows that G_n is not self-cyclising for all $n \in \mathbb{N}$, $n \geq 3$.

3.3 The generators of Cyc(G_n), for $n \geq 3$

We turn our attention to Cyc(G_n). In order to understand this group, we need to understand the cycles involved in elements of G_n. Since all the elements of G_n are bounded, it follows that all the cycles involved in elements of G_n are also bounded. We consider a particular type of bounded cycle.

Definition 3.8. *Let c be a bounded cycle in Sym(V_n). We say that c is an ultimately elementary cycle if c is finite, or*

$$c = (\cdots, v_{r,c+2a}, v_{r,c+a}, v_{r,c}, b_1, b_2, \cdots, b_k, v_{s,d}, v_{s,d+b}, v_{s,d+2b}, \cdots)$$

where $k \in \mathbb{N}$, $b_1, b_2, \cdots, b_k \in V_n$, r, s are distinct elements of $\{1, 2, \cdots, n\}$ and $a, b, c, d \in \mathbb{N}$.

All of the ultimately elementary cycles are elements of Cyc(G_n), as we shall see below.

Theorem 3.1. *Let c be an ultimately elementary cycle. Then c is involved in an element of G_n.*

Proof. If c is a finite cycle then the proof is straightforward: Lemma 3.1 implies that FS(V_n) is a subgroup of G_n, so c is an element of G_n. Every cycle is involved in itself, so c is involved in an element of G_n.

Assume c is infinite, so

$$c = (\cdots, v_{r,c+2a}, v_{r,c+a}, v_{r,c}, b_1, b_2, \cdots, b_k, v_{s,d}, v_{s,d+b}, v_{s,d+2b}, \cdots)$$

where r, s are distinct elements of $\{1, 2, \cdots, n\}$, a, b, c, d, k are natural numbers and $b_1, b_2, \cdots, b_k$ are elements of V_n. The remainder of the proof is quite technical and requires construction of several complicated infinite cycles. To ease the burden on the reader, we intersperse the proof with a worked example to illustrate what is taking place.

As noted in Remark 3.1, the permutations $\sigma_{k,l}$ are elements of G_n for all $k,l \in \{1,2,\cdots,n\}$ and so the permutation $h = \sigma_{r,s}^{a}\sigma_{t,s}^{b-a}$, where $t \in \{1,2,\cdots,n\}$ distinct from r and s, is an element of G_n. Furthermore, the cycle

$$\alpha = (\cdots, v_{r,1+2a}, v_{r,1+a}, v_{r,1}, v_{s,a-1}, v_{s,a-1+b}, v_{s,a-1+2b}, \cdots)$$

is involved in h. This cycle α moves elements of X_r in its support by translation by a towards the origin, and elements of X_s by translation by b away from the origin, which is also what c does.

Example 3.2. *Let $n = 3$ and let $c = (\cdots, v_{1,18}, v_{1,13}, v_{1,8}, O, v_{2,4}, v_{2,7}, v_{2,10}, \cdots)$. Using our earlier notation, we have $r = 1$, $s = 2$, $a = 5$ and $b = 3$. In our search to find an element of G_n that c is involved in, we observe that the permutation $h = \sigma_{1,2}^{5}\sigma_{3,2}^{-2}$ is an element of G_n. Furthermore,*

$$\begin{aligned} h =& (\cdots, v_{1,10}, v_{1,5}, O, v_{3,2}, v_{3,4}, \cdots) \\ & \circ (\cdots, v_{1,11}, v_{1,6}, v_{1,1}, v_{2,4}, v_{2,7}, v_{2,10} \cdots) \\ & \circ (\cdots, v_{1,12}, v_{1,7}, v_{1,2}, v_{2,3}, v_{2,6}, v_{2,9} \cdots) \\ & \circ (\cdots, v_{1,13}, v_{1,8}, v_{1,3}, v_{2,2}, v_{2,5}, v_{2,8} \cdots) \\ & \circ (\cdots, v_{1,14}, v_{1,9}, v_{1,4}, v_{2,1}, v_{3,1}, v_{3,3}, v_{3,5}, \cdots) \end{aligned}$$

and so the cycle

$$\alpha = (\cdots, v_{1,11}, v_{1,6}, v_{1,1}, v_{2,4}, v_{2,7}, v_{2,10}, \cdots)$$

is involved in h, an element of G_n.

Returning to the general analysis, notice that c and α need not have the same support on either of the branches X_r or X_s. We can fix this, up to a finite number of vertices, by conjugating h by suitable powers of $\sigma_{r,t}$ and $\sigma_{t,s}$. Let $f = \sigma_{r,t}^{1-c}$ and $g = \sigma_{t,s}^{d-a+1}$. Then h^{gf} is still an element of G_n, and furthermore the cycle

$$\alpha' = \alpha^{gf} = (\cdots, v_{r,c+2a}, v_{r,c+a}, v_{r,c}, v_{s,d}, v_{s,d+b}, v_{s,d+2b}, \cdots)$$

is involved in h^{gf} and differs from c on only a finite number of vertices.

Example 3.3. *Consider the cycle c and the permutation h as in Example 3.2. Using our earlier notation, we have $a - 1 = 4 = d$ and $c = 8$, so $f = \sigma_{1,3}^{-7}$ and $g = \sigma_{3,2}^{0} = id$. Then*

$$\begin{aligned} h^{gf} =& (\cdots, v_{1,17}, v_{1,12}, v_{1,7}, v_{1,5}, v_{1,3}, v_{1,1}, v_{3,1}, v_{3,3}, \cdots) \\ &\circ (\cdots, v_{1,18}, v_{1,13}, v_{1,8}, v_{2,4}, v_{2,7}, v_{2,10}, \cdots) \\ &\circ (\cdots, v_{1,19}, v_{1,14}, v_{1,9}, v_{2,3}, v_{2,6}, v_{2,9}, \cdots) \\ &\circ (\cdots, v_{1,20}, v_{1,15}, v_{1,10}, v_{2,2}, v_{2,5}, v_{2,8}, \cdots) \\ &\circ (\cdots, v_{1,21}, v_{1,16}, v_{1,11}, v_{2,1}, v_{1,6}, v_{1,4}, v_{1,2}, O, v_{3,2}, v_{3,4}, \cdots) \end{aligned}$$

and the cycle $\alpha' = (\cdots, v_{1,18}, v_{1,13}, v_{1,8}, v_{2,4}, v_{2,7}, v_{2,10}, \cdots)$ is involved in h and disagrees with c only on how it moves $v_{1,8}$ and O.

Once again returning to the general analysis, all that remains is to tidy up the finite discrepancies. Write h^{gf} as a product of disjoint cycles, so $h^{gf} = c_1 c_2 \cdots c_m \alpha'$ for some natural number m and cycles $c_1, c_2, \cdots, c_m$. Assume that $c_1 = (\cdots, a_{-1}, a_0, a_1, \cdots)$ and let

$$\delta = (v_{r,c}, a_0, a_k)$$

It follows that $\varphi := h^{gf}\delta \in G_n$ and

$$\varphi = (\cdots, a_{-2}, a_{-1}, a_0, a_{k+1}, a_{k+2}, \cdots) c_2 c_3 \cdots c_m \alpha''$$

where $\alpha'' = (\cdots, v_{r,c+a}, v_{r,c}, a_1, a_2, \cdots, a_k, v_{s,d}, v_{s,d+b}, \cdots)$. Now let

$$\varepsilon = \prod_{i=1}^{k} (a_i, b_i)$$

It follows that $\varphi^{\varepsilon} \in G_n$, and furthermore the cycle

$$(\alpha'')^{\varepsilon} = (\cdots, v_{r,c+2a}, v_{r,c+a}, v_{r,c}, b_1, b_2, \cdots, b_k, v_{s,d}, v_{s,d+b}, v_{s,d+2b}, \cdots) = c$$

is involved in φ^{ε}, an element of G_n. $\square$

Example 3.4. *Consider the cycle c and permutation h^{gf} as in Example 3.3. Using our earlier notation, $k = 1$ and $c_1 = (\cdots, v_{1,12}, v_{1,7}, v_{1,5}, v_{1,3}, v_{1,1}, v_{3,1}, v_{3,3}, \cdots)$. Let $\delta = (v_{1,8}, v_{1,3}, v_{1,1})$ and $\varepsilon = (v_{1,1}, O)$. Then*

$$\begin{aligned}\varphi = h^{gf}\delta =& (\cdots, v_{1,17}, v_{1,12}, v_{1,7}, v_{1,5}, v_{1,3}, v_{3,1}, v_{3,3}, v_{3,5}, \cdots) \\ &\circ (\cdots, v_{1,19}, v_{1,14}, v_{1,9}, v_{2,3}, v_{2,6}, v_{2,9}, \cdots) \\ &\circ (\cdots, v_{1,20}, v_{1,15}, v_{1,10}, v_{2,2}, v_{2,5}, v_{2,8}, \cdots) \\ &\circ (\cdots, v_{1,21}, v_{1,16}, v_{1,11}, v_{2,1}, v_{1,6}, v_{1,4}, v_{1,2}, O, v_{3,2}, v_{3,4}, \cdots) \\ &\circ \alpha''\end{aligned}$$

where $\alpha'' = (\cdots, v_{1,18}, v_{1,13}, v_{1,8}, v_{1,1}, v_{2,4}, v_{2,7}, v_{2,10}, \cdots)$. Thus, the cycle $(\alpha'')^{\varepsilon} = (\cdots, v_{1,18}, v_{1,13}, v_{1,8}, O, v_{2,4}, v_{2,7}, v_{2,10}, \cdots) = c$ is involved in φ^{ε}.

We now show that the converse statement is true; that every cycle involved in an element of G_n is an ultimately elementary cycle.

Theorem 3.2. *Let f be an element of G_n, and suppose that c is a cycle involved in f. Then c is an ultimately elementary cycle.*

Proof. Let $a_1, a_2, \cdots, a_n$ be the respective branch speeds for f as in Definition 3.6, and let N_0 be the minimal radius of non-uniformity for f. If c is finite, then by definition c is an ultimately elementary cycle, so assume that c is an infinite cycle. Since c is a bounded cycle, there exists $L \in \mathbb{N}$ such that for all $v \in V_n$, $|c(v) - v| < L$. Let

$$M := \max\{L, N_0\}$$

and let

$$\Omega = \{v_{r,x} \in V_n : x \leq M\}$$

We observe two things about Ω; firstly, Ω contains all the vertices $v_{r,x}$ of V_n which are moved by f in a way that differs from the ultimate action of f on the branch X_r, since Ω contains the ball of non-uniformity for f. Secondly, if $v_{r,x} \notin \Omega$, then $c(v_{r,x}) \notin X_s \setminus \Omega$ for any $s \in \{1, 2, \cdots, n\}$, $s \neq r$, because the maximum distance f (and therefore c) can move elements of V_n is less than the radius of Ω. Furthermore, observe that Ω is a finite set.

We may assume, without loss of generality, that there exists $v = v_{r,x} \in$ supp(c)

such that $v \notin \Omega$ and the speed of f on branch r is negative, i.e $a_r < 0$, for the following reasons: it's clear that we may choose $v \in \text{supp}(c) \setminus \Omega$ because Ω is a finite set. If all the elements in $\text{supp}(c) \setminus \Omega$ are on branches where f has a positive speed, work with f^{-1} instead. Inverting f changes the sign of all the branch speeds and inverts the cycles in its cycle decomposition, so c^{-1} is involved in f. It follows that $\text{supp}(c^{-1})$ must contain an element of the desired form (since the branch speed's signs have changed), and proving that c^{-1} is ultimately elementary implies that c is also ultimately elementary.

So assume that there exists $v = v_{r,x} \in \text{supp}(c)$ such that $v \notin \Omega$ and $a_r < 0$. We want to show that some power of c moves v to a vertex not in Ω, on a branch where the ultimate action of f is translation away from the origin. In other words, we want to show that for some $k \in \mathbb{N}$, $c^k(v) = v_{s,y}$ where $s \neq r$, $y > M$ and $a_s > 0$. If we can show this, then it follows that c must take the form

$$c = (\cdots, v_{r,x-a_r}, v_{r,x}, b_1, b_2, \cdots b_k, v_{s,y}, v_{s,y+a_s}, \cdots)$$

for some $b_1, b_2, \cdots, b_k \in V_n$, and so c takes the form of an ultimately elementary cycle. This is because both $v_{s,y}$ and $v_{r,x}$ are not in the set Ω. The map f has a positive speed on branch X_s and so $c(v_{s,y}) = f(v_{s,y}) = v_{s,y+a_s}$. Now $v_{s,y+a_s}$ is also not in Ω and so we know that f (and hence c) moves this vertex a distance of a_s away from the origin. This process continues indefinitely, and so the 'right tail' of c is determined. A similar line of thinking determines the 'left tail' of c, except that because the speed of f on branch r is negative, the ultimate action of f on branch r determines which vertex is mapped to $v_{r,x}$ by c, rather than to which vertex c maps $v_{r,x}$. The assumption that $c^k(v_{r,x}) = v_{s,y}$ explains the remaining 'central part' of c.

So assume that for all $k \in \mathbb{N}$, $c^k(v)$ is either in Ω, or is in $X_t \setminus \Omega$ for some $t \in \{1, 2, \cdots, n\}$ where $a_t < 0$. Notice that applying c to a vertex in $\text{supp}(c) \setminus \Omega$ translates that vertex in a way determined by the respective branch speed of f. In particular, v is translated $|a_r|$ places in the direction of the origin. Since there are only finitely many vertices in Γ_n that are between v and Ω and $|a_r| > 0$, there exists $k \in \mathbb{N}$ such that $c^k(v) \notin X_r \setminus \Omega$. Since the radius of Ω is greater than the maximum distance c moves any element of V_n, it must be that $c^k(v) \in \Omega$. Now c is an infinite cycle and Ω is a finite set, so there exists $l \in \mathbb{N}$ such that $c^l(c^k(v)) = c^{k+l}(v) \notin \Omega$. By assumption, $c^{k+l}(v) = v_{t,z}$ for some $t \in \{1, 2, \cdots, n\}, z > M$ and $a_s < 0$. By the same reasoning as above, there exists $m \in \mathbb{N}$ such that $c^m(c^{k+l}(v)) \in \Omega$. Again by the reasoning above, some power of c moves v to a vertex not in Ω, on a branch with negative branch speed. We can see that this process of c moving into and out of Ω will continue indefinitely, but this contradicts the fact that Ω is a finite set. Hence our original assumption was incorrect, and there exists $k \in \mathbb{N}$ such that $c^k(v) = v_{s,y}$ where $s \in \{1, 2, \cdots, n\}, y > M$ and $a_s > 0$. Therefore c is an ultimately elementary cycle by the previous reasoning. □

3.4 The cyclizer of G_n, for $n \geq 3$

Theorems 3.1 and 3.2 tell us that $\mathrm{Cyc}(G_n)$ is generated by G_n, along with all the ultimately elementary cycles. What group is this? To understand $\mathrm{Cyc}(G_n)$ it will help to re-interpret the definition of an ultimately elementary permutation.

Definition 3.9. *Let Γ_n be a labelled n-branched star with vertex set V_n. For all distinct pairs $r, s \in \{1, 2, \cdots, n\}$ let $Y_{r,s}$ be the induced subgraph of Γ_n with vertex set $V(X_r) \cup V(X_s) \cup \{O\}$ (where the subscripts are counted modulo n). When $r \equiv s - 1$ mod n we write Y_s instead.*

For all distinct pairs $r, s \in \{1, 2, \cdots, n\}$, the graph $Y_{r,s}$ is graph isomorphic to Γ_2 via the bijection $\varphi_{r,s} : \mathbb{Z} \to Y_{r,s}$,

$$\varphi_{r,s}(z) = \begin{cases} v_{s,z} & z > 0 \\ O & z = 0 \\ v_{r,-z} & z < 0 \end{cases}$$

If $r \equiv s - 1 \bmod n$, then we write φ_s instead. Since $\varphi_{r,s}$ is a bijection for each distinct pair $r, s \in \{1, 2, \cdots, n\}$, if f is a permutation of $\mathbb{Z}$, then $f^{\varphi_{r,s}}$ is a permutation of $Y_{r,s}$. Recall that G_2 is the subgroup of $\mathrm{Sym}(\mathbb{Z})$ generated by $\sigma : \mathbb{Z} \to \mathbb{Z}$, $\sigma(z) = z + 1$ for all $z \in \mathbb{Z}$. These two observations motivate the following:

Definition 3.10. *Let $r, s \in \{1, 2, \cdots, n\}$ be distinct, and let g be a permutation of $Y_{r,s}$. We say that g is* **elementary** *on $Y_{r,s}$ if $g = f^{\sigma_{r,s}}$ for some $f \in G_2$.*

Now every element of G_2 is of the form $f : \mathbb{Z} \to \mathbb{Z},\ f(z) = z + a$ for some $a \in \mathbb{Z}$, and so a permutation $g \in \mathrm{Sym}(Y_{r,s})$ is elementary if it acts on $Y_{r,s}$ by translating every vertex of $Y_{r,s}$ by a fixed distance in the same direction.

Let f be an ultimately elementary permutation, with branch speeds $a_1, a_2, \cdots, a_n$ and radius of non-uniformity $N \in \mathbb{N}$. For each $s \in \{1, 2, \cdots, n\}$ let f_s be the element of G_2 such that

$$f_s : \mathbb{Z} \to \mathbb{Z}, f_s(z) = z + a_s$$

Observe that the ultimate action of f on branch s is identical to the elementary permutation $f_s^{\sigma_s} \in \mathrm{Sym}(Y_s)$; specifically,

$$\forall x > N,\ f(v_{s,x}) = f_s^{\sigma_s}(v_{s,x})$$

Conversely, if there exists $N \in \mathbb{N}$ and for all $s \in \{1, 2, \cdots, n\}$ there exists $f_s \in G_2$ such that $f(v_{s,x}) = f_s^{\sigma_s}(v_{s,x})$ for all $x > N$, then f is ultimately elementary, with radius of non-uniformity N and branch speeds $a_1, a_2, \cdots a_n$, where $f_s(z) = z + a_s$ for all $z \in \mathbb{Z}$. It follows from this that the definition below is an equivalent definition of an ultimately elementary permutation.

Definition 3.11. *Let f be a permutation of V_n. We say that f is* **ultimately elementary** *if there is $N \in \mathbb{N}$ and there is $f_s \in G_2$ for each $s \in \{1, 2, \cdots, n\}$ such that*

$$f(v_{s,x}) = f_s^{\sigma_s}(v_{s,x}) \forall x > N$$

Recalling the theory for the cyclizer chain of G_2 in [8], $\mathrm{Cyc}(G_2)$ is generated by powers of $\sigma : \mathbb{Z} \to \mathbb{Z}, \sigma(z) = z + 1$ (the elements of G_2) and the elementary permutations (the cycles involved in elements of G_2); this group is the group of modular permutations of G_2. In the case where $n \geq 3$, G_n is the group of permutations whose ultimate actions on a branch of G_n are identical to some power of σ, and the cycles involved in these elements are cycles whose ultimate actions on a branch of G_n are identical to an elementary permutation of $\mathbb{Z}$. These observations suggest that G_{mod}, the modular permutations of the integers, might be related to the cyclizer series of G, and motivates the following definition.

Definition 3.12. *Let $s \in \{1, 2, \cdots, n\}$ and let g be a permutation of Y_s. We say that g is* **modular on** Y_s *if $g = f^{\varphi_s}$ for $f \in G_{\mathrm{mod}}$.*

This definition allows us to generalise the concept of a modular permutation to the group G_n.

Definition 3.13. *Let f be a bounded permutation of V_n. We say that f is* **ultimately modular** *on V_n if there exists $N > 0$ and $f_1, f_2, \cdots, f_n \in G_{\mathrm{mod}}$ such that for all $x > N$,*

$$f(v_{s,x}) = v_{s,f_s(x)}$$

For each $s \in \{1, 2, \cdots n\}$ we call f_s the **branch action** *of f on X_s, and we call N a* **radius of non-modularity** *for f.*

In other words, a bounded permutation f of V_n is ultimately modular if the action of f on branch X_s is identical almost everywhere to the action of a modular permutation of Y_s. Notice that an ultimately modular permutation of V_2 is a permutation with modular ends, in the sense of Fiddes' thesis [8]. This observation will be relevant for section 4.2.

If f is ultimately modular on V_n with branch actions $f_1, f_2, \cdots, f_n$ respectively, then by definition there exists $m_1, m_2, \cdots, m_n \in \mathbb{Z}$ such that $f_s(x + m_s) = f_s(x) + m_s$ for all $x \in \mathbb{Z}$. It follows that for all $x > N$,

$$f(v_{s,x+m_s}) = v_{s,f_s(x)+m_s}$$

Also note that if $f \in \mathrm{Sym}(V_n)$ is ultimately elementary then f is ultimately modular, because G_{mod} is the cycliser of G_2, and therefore contains G_2. The ultimate action of f on any branch of Γ_n is identical to some element of G_2, so the ultimate action is identical to some element of G_{mod}, which means f is ultimately modular on V_n.

Denote the set of all ultimately modular permutations of V_n by M_n. As might be expected, M_n is a group for each $n \geq 3$.

Lemma 3.3. *M_n is a group for each $n \in \mathbb{N}$, $n \geq 3$.*

Proof. Firstly, id_{V_n} is an ultimately modular permutation, with all branch actions equal to $id_{\mathbb{Z}}$ and with radius of non-modularity 0. Secondly, if f is an ultimately modular permutation with branch actions $f_1, f_2, \cdots, f_n$ and radius of non-modularity N, then f^{-1} is ultimately modular with branch actions $f_1^{-1}, f_2^{-1}, \cdots, f_n^{-1}$ and radius of non-modularity N. Finally, if g is also an ultimately modular permutation, with branch actions $g_1, g_2, \cdots g_n$ and radius of non-modularity N', then $f \circ g$ is ultimately modular with branch actions $f_1 \circ g_1, f_2 \circ g_2, \cdots, f_n \circ g_n$ and radius of non-modularity $\max\{N, N'\}$. □

The following two results show that $\mathrm{Cyc}(G_n) = M_n$, confirming our suspicions that the modular permutations of $\mathbb{Z}$ are related to the cyclizer series of G_n, and reminding us of the result of Cameron, that $\mathrm{Cyc}^3(\mathbb{Z})$ is the group of permutations with modular ends; in fact we can view the following two lemmas as a generalisation of this result of Cameron.

Lemma 3.4. *$Cyc(G_n) \leq M_n$*

Proof. It suffices to show that all the generators of $\mathrm{Cyc}(G_n)$ are ultimately modular. Recall that $\mathrm{Cyc}(G_n)$ is generated by the ultimately elementary permutations that satisfy the Kirchoff property, along with the ultimately elementary cycles. As we observed above, every ultimately elementary permutation is ultimately modular, so we just have to show that the ultimately elementary cycles are ultimately modular. The ultimate action of an ultimately elementary cycle is identical to either $id_{\mathbb{Z}}$ or an elementary permutation of $\mathbb{Z}$. Both $id_{\mathbb{Z}}$ and the elementary permutations are contained in G_{mod}, and so an ultimately elementary cycle is ultimately modular. Hence $\mathrm{Cyc}(G_n) \leq M_n$. □

For the next result it will be necessary to generalise Remark 3.2 for the ultimately modular permutations. Let f be an ultimately modular permutation. Since all the radii of non-modularity for f are natural numbers, there is a minimal radius of non-modularity: call it N_0. Define the set

$$\Omega = \{v_{r,x} \in V_n | x \leq N_0\}$$

As in the ultimately elementary case, we are describing the smallest ball around the origin of Γ_n, such that every vertex outside this ball is moved by f according to the respective branch action, and then let Ω be the vertices of Γ_n inside this ball. As in Remark 3.2, we call Ω the **ball of non-modularity**.

Now we can prove the reverse containment of Lemma 3.4.

Lemma 3.5. *$M_n \leq Cyc(G_n)$*

Proof. Let f be a modular permutation of V_n, with branch actions $f_1, f_2, \cdots f_n$ and radius of non-modularity N. We first consider the case where $f_2 = f_3 = \cdots f_n = id_{\mathbb{Z}}$. In this case, consider the modular permutation $f_1 \in G_{\text{mod}}$. We show that f_1 has zero flow. Recall that a permutation $g \in \text{Sym}(\mathbb{Z})$ has zero flow if the net flow $\phi_x(g) = 0$ for some $x \in \mathbb{Z} + \frac{1}{2}$. Assume for contradiction that f_1 does not have zero flow, and let $x \in \mathbb{Z} + \frac{1}{2}$. Without loss of generality, assume that $\phi_x^-(f_1) > 0$; if not, work with f_1^{-1} instead as if f_1^{-1} has zero flow, then f_1 also has zero flow. Let Ω be the ball of non-modularity for f, and let z be an element of the set $\{y \in \mathbb{Z} : y > x, f_1(y) < x\}$. Notice that there is no $k \in \mathbb{N}$ such that $f_1^k(z) > x$, because if there were then $\phi_x^+(f_1) = \phi_x^-(f_1)$, which implies $\phi_x(f_1) = 0$, which contradicts our original assumption. Let

$$\Delta = \{v_{1,y} : M < y < x\}$$

Since Δ is finite, there exists $k \in \mathbb{N}$ such that $f^k(v_{1,z}) \notin X_1 \setminus \Omega$. Since the only branch action of f that is not $id_{\mathbb{Z}}$ is f_1, it follows that $f^k(v_{1,z}) \in \Omega$. Now Ω is a finite set and so there exists $l \in \mathbb{N}$ such that $f^{k+l}(v_{1,z}) \notin \Omega$. The only possibility is that $f^{k+l}(v_{1,z}) \in \Delta$, since all the branch actions of f except for f_1 are $id_{\mathbb{Z}}$, and $\phi_x(f_1) \neq 0$. It's clear that this process of moving between Ω and Δ will continue indefinitely, as f_1 is an infinite permutation. However, this contradicts the finiteness of Δ and Ω, and so f_1 must have zero flow, i.e. $f_1 \in G_{\text{mod}}$.

Theorem 2.3 tell us that G_{mod} is simple, and so it follows that G_{mod} is perfect, i.e. we can write every element of G_{mod} as a product of commutators in G_{mod}. Therefore we can write

$$f_1 = \prod_{i=1}^{k} [a_i, b_i]$$

where $k \in \mathbb{N}$ and the permutations a_i and b_i are elements of G_{mod} for each $i \in \{1, 2, \cdots, k\}$. We will use this decomposition to show that $f \in \text{Cyc}(G_n)$. Notice that if $g \in G_{\text{mod}}$ then g can be written as a product of elementary permutations of $\mathbb{Z}$, i.e.

$$g = e_1 e_2 \cdots e_m$$

where $m \in \mathbb{N}$ and $e_1, e_2, \cdots, e_m$ are elementary. Furthermore, if e is an elementary permutation of $\mathbb{Z}$ and $r, s \in \{1, 2, \cdots, n\}$ are distinct then, considering it as permutation of V_n, $e^{\phi_{r,s}}$ is an ultimately elementary cycle. Therefore, when considered as a permutation of V_n,

$$g^{\phi_{r,s}} = e_1^{\phi_{r,s}} e_2^{\phi_{r,s}} \cdots e_m^{\phi_{r,s}}$$

is a product of ultimately elementary cycles and hence is an element of $\text{Cyc}(G_n)$. In particular, this means that for $k \in \mathbb{N}$ the permutation

$$f' := \prod_{i=1}^{k} [a_i^{\phi_1}, b_i^{\phi_{2,1}}]$$

is an element of $\text{Cyc}(G_n)$. Now f' agrees with f on all but a finite number of vertices and so we can write $f = f' \circ \varepsilon$ for some $\varepsilon \in FS(V_n)$. We have seen that $f' \in \text{Cyc}(G_n)$ and since $FS(V_n) \trianglelefteq G_n \leq \text{Cyc}(G_n)$ we have that $\varepsilon \in \text{Cyc}(G_n)$, hence $f \in \text{Cyc}(G_n)$.

Now let $f \in M_n$, i.e. f is an ultimately modular permutation of V_n. Assume the branch actions of f are $f_1, f_2, \cdots, f_n$. We shall exhibit a decomposition of f into a product of elements of $\text{Cyc}(G_n)$, using the above arguments to help do so. Consider the permutation $(f_n^{-1})^{\varphi_n} \in \text{Sym}(Y_n)$. Since f_n is a modular permutation of $\mathbb{Z}$ it follows that $(f_n^{-1})^{\varphi_n}$ is ultimately modular, when we consider it a permutation of V_n. Let $f_n' := (f_n^{-1})^{\varphi_n}$. Then $f \circ f_n'$ is also ultimately modular and the branch actions of $f \circ f_n'$ are $f_1, f_2, \cdots, f_{n-2}, g_{n-1}, id_{\mathbb{Z}}$ for some $g_{n-1} \in G_{\text{mod}}$.

Now consider $(g_{n-1}^{-1})^{\varphi_{n-1}}$; considered as a permutation of V_n this is an ultimately modular permutation. If we define $f_{n-1}' := (g_{n-1}^{-1})^{\varphi_{n-1}}$ then $f \circ f_n' \circ f_{n-1}' \in M_n$ and the branch actions of this permutation are $f_1, f_2, \cdots, f_{n-2}, g_{n-2}, id_{\mathbb{Z}}, id_{\mathbb{Z}}$ for some $g_{n-2} \in G_{\text{mod}}$. It follows that we may continue this process until we construct an element $h := f \circ f'_n \circ f'_{n-1} \circ \cdots \circ f'_2 \in M_n$ such that $f'_i \in M_n$ for each $i \in \{2, 3, \cdots, n\}$ and the branch actions of h are $h_1, id_{\mathbb{Z}}, id_{\mathbb{Z}}, \cdots, id_{\mathbb{Z}}$ for some $h_1 \in M_n$. Therefore h is a permutation of the form discussed above, and so using those arguments we know that $h \in \text{Cyc}(G_n)$. It follows that

$$f = h \circ (f'_2)^{-1} \circ (f'_3)^{-1} \circ \cdots \circ (f'_n)^{-1} \in \text{Cyc}(G_n)$$

and so we are done. □

Corollary 3.2. *Cyc(G_n) is the group of ultimately modular permutations of V_n.*

Now that we have determined $\text{Cyc}(G_n)$, the sensible next step is to determine $\text{Cyc}^2(G_n)$. To do this we need to understand the cycles involved in $\text{Cyc}(G_n)$, i.e. the cycles involved in ultimately modular permutations of V_n. It will be useful during this analysis to have a tidy way of writing down an infinite cycle of V_n. If c is such a cycle, we can always write it in the form

$$c = (\cdots, v_{-2}, v_{-1}, v_0, v_1, v_2, \cdots)$$

where v_0 is any element in $\text{supp}(c)$, and $v_i := c^i(v_0)$ for each non-negative integer i. The following result is a useful property of infinite cycles involved in bounded permutations of V_n.

Proposition 3.2. *Let f be a bounded permutation of V_n, and let c be an infinite cycle involved in f. Then there exists $r, s \in \{1, 2, \cdots, n\}$ such that*

$$c = (\cdots, v_{r,x_3}, v_{r,x_2}, v_{r,x_1}, v_i, v_{i+1}, \cdots, v_{j-1}, v_j, v_{s,y_1}, v_{s,y_2}, v_{s,y_3}, \cdots)$$

for some integers $x_1, x_2, \cdots$ and $y_1, y_2, \cdots$ and where $v_i, v_{i+1}, \cdots, v_j$ are vertices of Γ_n.

Proof. Write c in the form $(\cdots, v_{-2}, v_{-1}, v_0, v_1, v_2, \cdots)$ as outlined above. Note that $c(v) = f(v)$ for all $v \in \text{supp}(c)$ since c is involved in f, and so $v_i = f^i(v_0)$ for all integers i. Therefore c must be bounded, because if it was not then f would not be bounded either. Let M be a bound for c. Since c is infinite, the Dirichlet principle tell us that its support must have infinite intersection with one of the branches of Γ_n; assume this branch is X_s. It follows from the boundedness of c that if $v = v_{s,x} \in V(X_s) \cap \mathrm{supp}(c)$ with $x > M$, that $c(v) \in v(X_s) \cap \mathrm{supp}(c)$ also. All but finitely many elements of $V(X_s) \cap \mathrm{supp}(c)$ satisfy this condition, and so the sequence $(c^k(v_0))_{k=-\infty}^{\infty}$ contains infinitely many consecutive terms from $v(X_s) \cap \mathrm{supp}(c)$. This means that either the 'right tail', 'left tail' or 'both tails' of c are exclusively vertices from $V(X_s) \cap \mathrm{supp}(c)$, i.e. either c takes the form

$$(\cdots, v_{s,x_3}, v_{s,x_2}, v_{s,x_1}, v_i, v_{i+1}, v_{i+2}, \cdots)$$

for some $i \in \mathbb{Z}$; or c takes the form

$$(\cdots, v_{j-2}, v_{j-1}, v_j, v_{s,y_1}, v_{s,y_2}, v_{s,y_3}, \cdots)$$

for some $j \in \mathbb{Z}$; or c takes the form

$$(\cdots, v_{s,x_2}, v_{s,x_1}, v_i, v_{i+1}, \cdots v_{j-1}, v_j, v_{s,y_1}, v_{s,y_2}, \cdots)$$

for some $i, j \in \mathbb{Z}$, $i \leq j$. If c is of the last form then the result is proven. If c is of one of the first two forms, notice that the relevant 'tail' contains all but finitely many elements of $V(X_s) \cap \mathrm{supp}(c)$, and so by the Dirichlet principle there is a branch X_r where $r \neq s$ such that $V(X_r) \cap \mathrm{supp}(c)$ is infinite. Using the boundedness of c and the same logic as above we deduce that the sequence $(c^k(v_0))_{k=-\infty}^{\infty}$ contains infinitely many consecutive terms from $V(X_r) \cap \mathrm{supp}(c)$, and so c must either be of the form

$$(\cdots, v_{r,x_2}, v_{r,x_1}, v_i, v_{i+1}, \cdots v_{j-1}, v_j, v_{s,y_1}, v_{s,y_2}, \cdots)$$

or be of the form

$$(\cdots, v_{s,x_2}, v_{s,x_1}, v_i, v_{i+1}, \cdots v_{j-1}, v_j, v_{r,y_1}, v_{r,y_2}, \cdots)$$

for some $i, j \in \mathbb{Z}$, $i \leq j$, which proves the result. □

Notice that every modular permutation is bounded, because the set of distances from v to $f(v)$, where v ranges across all of V_n, is finite. If f is a ultimately modular permutation of V_n and c is an infinite cycle involved in f then c is bounded, because $c(v) = f(v)$ for all $v \in \text{supp}(c)$ and so if c was not bounded then f would not be bounded either. This next result is in some sense a generalisation of Propostion 2.2.

Proposition 3.3. *Let f be an ultimately modular permutation of V_n, and suppose that c is an infinite cycle involved in f. Then c is an ultimately modular permutation of V_n.*

Proof. As in Proposition 3.2, let $v_0 \in \text{supp}(c)$ and for each non-negative integer i let $v_i = c^i(v_0)$, so $c = (\cdots, v_{-2}, v_{-1}, v_0, v_1, v_2, \cdots)$. Proposition 3.2 implies that we can write c in the form

$$(\cdots, v_{r,x_3}, v_{r,x_2}, v_{r,x_1}, v_i, v_{i+1}, \cdots, v_{j-1}, v_j, v_{s,y_1}, v_{s,y_2}, v_{s,y_3}, \cdots)$$

for some $r, s \in \{1, 2, \cdots, n\}$ and integers i, j with $i \leq j$. Assume that the branch action of f is $(f_1, f_2, \cdots, f_n)$ and the modulus of f_i is m_i, for each $i \in \{1, 2, \cdots, n\}$. We deal first with the case where $r \neq s$. By the Dirichlet principle, there exists $a, b \in \mathbb{Z}$ such that $a < b$, $y_b \equiv y_a \mod m_s$ and $c := b - a$ is minimal. Similarly, there exists $\alpha, \beta \in \mathbb{Z}$ such that $\alpha < \beta$, $x_\beta \equiv x_\alpha \mod m_r$ and $\gamma := \beta - \alpha$ is minimal. Since y_a and y_b are congruent modulo m_s, there exists $l \in \mathbb{Z}$ such that $y_b = y_a + lm_j$. Similarly, there exists $k \in \mathbb{Z}$ such that $x_\beta = x_\alpha + km_r$. For notational convenience, define $p_r := km_r$ and $p_s := lm_s$. Define the following cycles of the integers:

$$c_r = (\cdots, x_{\gamma-1-p_r}, x_{\gamma-p_r}, x_1, x_2, \cdots, x_\gamma, x_{1+p_r}, x_{2+p_r}, \cdots)$$

and

$$c_s = (\cdots, y_{c-1-p_s}, y_{c-p_s}, y_1, y_2, \cdots, y_c, y_{1+p_s}, y_{2+p_s}, \cdots)$$

We want to show that c_r and c_s are modular permutations of $\mathbb{Z}$, then show that c is modular with branch actions c_r and c_s on the branches that c has infinite support. We can show that c_r and c_s are modular on $\mathbb{Z}$ directly, by doing long-hand calculations to check by definition that c_r and c_s have moduli p_r and p_s respectively, but there is a nicer way. Considering c_r first, notice that c_r is a cycle involved in the branch action f_r: There is a cycle involved in f_r that has an identical 'right tail' to c_r. Since f_r is modular on $\mathbb{Z}$, the remaining behaviour on the 'left tail' of this cycle is determined by the modular structure of f_r. The cycle

c_r is the 'extension' of this modular structure onto the 'left tail', and agrees with this cycle in question on the 'right tail', so this cycle is c_r. Similarly, c_s is a cycle involved in the branch action f_s. Since f_r and f_s are modular permutations of $\mathbb{Z}$, Proposition 2.2 implies that c_r and c_s are modular on $\mathbb{Z}$. Furthermore, it's clear that for all $z > y_1$ that $c(v_{s,z}) = v_{s,c_s(z)}$ and for all $z > x_1$ that $c(v_{r,z}) = v_{r,c_r(z)}$ and so c is modular on V_n. It remains to deal with the case $r = s$. In this case, define

$$\begin{aligned}
c' &= c_r^{-1} \circ c_s \\
&= (\cdots, x_{2+p_r}, x_{1+p_r}, x_\gamma, x_{\gamma-1}, \cdots, x_1, x_{\gamma-p_r}, x_{\gamma-1-p_r}, \cdots) \\
&\quad \circ (\cdots, y_{c-1-p_s}, y_{c-p_s}, y_1, y_2, \cdots, y_c, y_{1+p_s}, y_{2+p_s}, \cdots)
\end{aligned}$$

We have already seen that c_r and c_s are modular on $\mathbb{Z}$ and so c' must also be modular on $\mathbb{Z}$. Furthermore it is straightforward to see that $c(v_{r,z}) = v_{v,c'(z)}$ for all $z > \max\{x_1, y_1\}$. Therefore c is modular on V_n and the result is proved. $\square$

Proposition 3.3 gives us an understanding of the infinite cycles involved in elements of $\mathrm{Cyc}(G_n)$, but what about the finite cycles? We already know from Lemma 3.1 that the group $FS(V_n)$ is a normal subgroup of G_n. Therefore all the finitary permutations of V_n are contained in $\mathrm{Cyc}(G_n)$, and so in particular all the finite cycles involved in elements of $\mathrm{Cyc}(G_n)$ are contained in $\mathrm{Cyc}(G_n)$. These two observations imply that every cycle involved in an element of $\mathrm{Cyc}(G_n)$ is also an element of $\mathrm{Cyc}(G_n)$, and so $\mathrm{Cyc}^2(G_n)$ is generated by precisely $\mathrm{Cyc}(G_n)$. Therefore, this means that:

Theorem 3.3. *$Cyc(G_n)$ is cycle-closed, i.e.*

$$Cyc^2(G_n) = Cyc(G_n)$$

This result is in agreement with the results about the cyclizer chain of $\mathbb{Z}$: In the case of $\mathbb{Z}$, the cyclizer chain ends at $\mathrm{Cyc}^3(G_n)$, which is the group of ultimately modular permutations of $\mathbb{Z}$. In the case of G_n, the cyclizer chain again ends at the group of ultimately modular permutations on G_n, but we reach this group two steps earlier than in the case of $\mathbb{Z}$.

Chapter 4

Additional theory about groups acting on n-branched stars

4.1 Flow on n-branched stars

Recall that when we studied the integers as a permutation group acting on itself in Chapter 2, we defined a property of elements of this group called the **flow** of an element; in essence the flow of of an element f is the net movement f causes in the positive direction. It turns out that there is a very natural generalisation of this concept to the group G_n acting on an n-branched star that we studied in Chapter 3.

Definition 4.1. *Let Γ_n be a labelled n-branched star with vertex set V_n and let $g \in Sym(V_n)$. For each $x \in \mathbb{N} + \frac{1}{2}$ and each $r \in \{1, 2, \cdots, n\}$ define*

$$\begin{aligned}
A_{r,x} &= \{v \in V_n : v = v_{r,y}, y > x\} \\
B_{r,x} &= V_n \setminus A_{r,x} \\
&= \{v \in V_n : v = v_{r,y}, y < x\} \\
&\qquad \cup \{v \in V_n : v = v_{s,y}, s \neq r\} \cup \{O\}
\end{aligned}$$

Then define

$$\phi^i_{r,x}(g) = |\{v \in V_n : v \in A_{r,x}, g(v) \in B_{r,x}\}|$$

$$\phi^o_{r,x}(g) = |\{v \in V_n : v \in B_{r,x}, g(v) \in A_{r,x}\}|$$

Definition 4.2. *Let $g \in Sym(V_n)$. If $\phi^i_{r,x}(g), \phi^o_{r,x}(g)$ are finite for all $r \in \{1, 2, \cdots, n\}$ and for all $x \in \mathbb{N} + \frac{1}{2}$, then we say that g is a* **finite flow permutation** *(or that g has* **finite flow***). If g has finite flow then we define $\phi_{r,x}(g) = \phi^o_{r,x}(g) - \phi^i_{r,x}(g)$, and we call $\phi_{r,x}(g)$ the* **net flow** *of g on the branch X_r.*

Next we show that flow is constant along any branch of V_n.

Lemma 4.1. *Let $g \in Sym(V_n)$. Let $r \in \{1, 2, \cdots, n\}$, let $x \in \mathbb{N}$ and assume $\phi^i_{r,x}(g)$ and $\phi^o_{r,x}(g)$ are finite, and that the net flow of g at x on X_r is p. Then $\forall y \in \mathbb{N} + \frac{1}{2}$, $\phi^i_{r,y}(g), \phi^o_{r,y}(g)$ are finite and the net flow of g at y on X_r is p.*

Proof. Let $y \in \mathbb{N} + \frac{1}{2}$. If $y = x$ there is nothing to prove, so firstly we assume that $y < x$. Define

$$S = \{v \in V_n : v = (r, z), y < z < x\}$$

Now S is a finite set and g is a bijection, so

$$|\{v \in S : g(v) \notin S\}| = |\{v \notin S : g(v) \in S\}| \tag{4.1}$$

Then

$$\begin{aligned}
\phi^o_{r,y}(g) &= |\{v \in B_{r,y} : g(v) \in A_{r,y}\}| \\
&= |\{v \in B_{r,y} : g(v) \in A_{r,x}\}| + |\{v \in B_{r,y} : g(v) \in S\}| \\
&= \underbrace{|\{v \in B_{r,x} : g(v) \in A_{r,x}\}|}_{\phi^o_{r,x}(g)} - |\{v \in S : g(v) \in A_{r,x}\}| + |\{v \in B_{r,y} : g(v) \in S\}| \\
&< \infty
\end{aligned}$$

since S is finite and g is a bijection. Similarly,

$$\begin{aligned}
\phi^i_{r,y}(g) &= |\{v \in A_{r,y} : g(v) \in B_{r,y}\}| \\
&= \phi^i_{r,x}(g) - |\{v \in A_{r,x} : g(v) \in S\}| + |\{v \in S : g(v) \in B_{r,y}\}| \\
&< \infty
\end{aligned}$$

since S is finite and g is a bijection. Furthermore,

$$\begin{aligned}
p &= \phi^o_{r,x}(g) - \phi^i_{r,x}(g) \\
&= \phi^o_{r,y}(g) + |\{v \in S : g(v) \in A_{r,x}\}| - |\{v \in B_{r,y} : g(v) \in S\}| \\
&\quad - \phi^i_{r,y}(g) - |\{v \in A_{r,x} : g(v) \in S\}| + |\{v \in S : g(v) \in B_{r,y}\}| \\
&= \phi^o_{r,y}(g) - \phi^i_{r,y}(g) + |\{v \in S : g(v) \notin S\}| - |\{v \notin S : g(v) \in S\}| \\
&= \phi^o_{r,y}(g) - \phi^i_{r,y}(g)
\end{aligned}$$

since $V_n = A_{r,x} \cup B_{r,y} \cup S$, and because of Equation 4.1. The argument for $y > x$ is similar, with the roles of x and y reversed. $\square$

It therefore makes sense to talk about the flow of g on an entire branch. Define the net flow on X_r, $\phi_r(g) = \phi^o_{r,x}(g) - \phi^i_{r,x}(g)$ for any $x \in \mathbb{N} + \frac{1}{2}$. Let FF_n be the set of finite flow permutations of V_n.

Definition 4.3. *We define the map $\phi : FF_n \to \mathbb{Z}^n$, where for all $g \in FF_n$,*

$$\phi(g) = (\phi_1(g), \phi_2(g), \ldots, \phi_n(g))$$

Lemma 4.2. $\phi(g \circ f) = \phi(f) + \phi(g)$

Proof. Let $f, g \in FF_n$ and let $r \in \{1, 2, \cdots, n\}$. We show that $\phi_r(g \circ f) = \phi_r(f) + \phi_r(g)$. Now

$$
\begin{aligned}
\phi^o_{r,\frac{1}{2}}(g\circ f) &= |\{v\in V_n : v\in B_{r,\frac{1}{2}}, g\circ f(v)\in A_{r,\frac{1}{2}}\}|\\
&= |\{v\in V_n : v\in B_{r,\frac{1}{2}}, f(v)\in A_{r,\frac{1}{2}}, g\circ f(v)\in A_{r,\frac{1}{2}}\}|\\
&\quad + |\{v\in V_n : v\in B_{r,\frac{1}{2}}, f(v)\in B_{r,\frac{1}{2}}, g\circ f(v)\in A_{r,\frac{1}{2}}\}|\\
&= \underbrace{|\{v\in V_n : v\in B_{r,\frac{1}{2}}, f(v)\in A_{r,\frac{1}{2}}\}|}_{\phi^o_{r,\frac{1}{2}}(f)}\\
&\quad - \underbrace{|\{v\in V_n : v\in B_{r,\frac{1}{2}}, f(v)\in A_{r,\frac{1}{2}}, g\circ f(v)\in B_{r,\frac{1}{2}}\}|}_{c_1}\\
&\qquad + |\{v\in V_n : v\in B_{r,\frac{1}{2}}, f(v)\in B_{r,\frac{1}{2}}, g\circ f(v)\in A_{r,\frac{1}{2}}\}|\\
&= \phi^o_{r,\frac{1}{2}}(f) - c_1 + |\{v\in V_n : f^{-1}(v)\in B_{r,\frac{1}{2}}, v\in B_{r,\frac{1}{2}}, g(v)\in A_{r,\frac{1}{2}}\}|\\
&= \phi^o_{r,\frac{1}{2}}(f) - c_1 + \phi^o_{r,\frac{1}{2}}(g)\\
&\quad - |\{v\in V_n : f^{-1}(v)\in A_{r,\frac{1}{2}}, v\in B_{r,\frac{1}{2}}, g(v)\in A_{r,\frac{1}{2}}\}|\\
&= \phi^o_{r,\frac{1}{2}}(f) - c_1 + \phi^o_{r,\frac{1}{2}}(g)\\
&\quad - \underbrace{|\{v\in V_n : v\in A_{r,\frac{1}{2}}, f(v)\in B_{r,\frac{1}{2}}, g\circ f(v)\in A_{r,\frac{1}{2}}\}|}_{c_2}
\end{aligned}
$$

Similarly, $\phi^i_{r,\frac{1}{2}}(g\circ f) = \phi^i_{r,\frac{1}{2}}(f) - c_2 + \phi^i_{r,\frac{1}{2}}(g) - c_1$ and so

$$
\begin{aligned}
\phi_r(g\circ f) &= \phi^o_{r,\frac{1}{2}}(g\circ f) - \phi^i_{r,\frac{1}{2}}(g\circ f)\\
&= (\phi^o_{r,\frac{1}{2}}(f) - \phi^i_{r,\frac{1}{2}}(f)) + (\phi^o_{r,\frac{1}{2}}(g) - \phi^i_{r,\frac{1}{2}}(g))\\
&= \phi_r(f) + \phi_r(g)
\end{aligned}
$$

and hence $\phi(g\circ f) = \phi(f)+\phi(g)$. □

Theorem 4.1. $FF_n \leq \mathit{Sym}(V_n)$

Proof. Clearly id_X has finite flow because $\phi^o_r(id_X) = \phi^i_r(id_X) = 0\,\forall r\in\{1,2,\cdots,n\}$. If f,g have finite flow then Lemma 4.2 implies that $g\circ f$ also has finite flow (since $\phi_r(g\circ f) = \phi_r(f)+\phi_r(g)$ $\forall r\in\{1,2,\cdots,n\}$ and $\phi_r(f),\phi_r(g)$ are finite). Finally if f has finite flow, then $\phi^o_r(f^{-1}) = \phi^i_r(f)$ and $\phi^i_r(f^{-1}) = \phi^o_r(f)$ and so $\phi_r(f^{-1}) = -\phi_r(f)$. Hence f^{-1} has finite flow. □

Lemma 4.2 and Theorem 4.1 imply that the map ϕ is a homomorphism. Notice that FF_n is cycle-closed, since every cycle involved in an element of FF_n has finite flow on V_n.

4.2 Modular permutations on n-branched stars

Now that we have a generalisation of the concept of flow for permutations of V_n, we can consider what results about the flow of permutations of $\mathbb{Z}$ might generalise as well. Recall that Theorem 2.3 says that $G_{\text{mod}}(0)$, the group of zero flow, modular permutations of $\mathbb{Z}$ is simple. Can we generalise this to a result about permutations of X_n? Recall that G_{mod} is in fact $\text{Cyc}(\mathbb{Z})$, and so $G_{\text{mod}}(0)$ can also be thought of as the group of zero flow permutations in $\text{Cyc}(\mathbb{Z})$. Is it the case that the group of zero flow permutations in $\text{Cyc}(G_n)$ is simple?

It is clear that M_n, the group of ultimately modular permutations of V_n, is a subgroup of FF_n, because M_n is a subgroup of the group of bounded permutations of V_n, which in turn is a subgroup of FF_n. Thus the concept of the flow of an element of M_n makes sense; in particular, we have a subgroup of M_n consisting of the elements of M_n with zero flow (if you like, the kernel of the map $\phi|_{M_n} : M_n \to \mathbb{Z}_n$). We denote this group $M_n(0)$. Since $M_n = \text{Cyc}(G_n)$, the question we wish to know the answer to is actually this: is $M_n(0)$ simple?

This question is actually very easily answered in the negative, as $FS(V_n)$, the group of finitary permutations of V_n, is a normal subgroup of $M_n(0)$ by Lemma 3.1. The next question would be: how 'close' to being simple is $M_n(0)$? To answer this, we investigate the quotient group $M_n(0)/FS(V_n)$.

Theorem 4.2. *$M_n(0)/FS(V_n) \cong \prod_{i=1}^{n-1} G_{\text{mod}}(0)$, for all $n \in \mathbb{N}$, $n \geq 3$.*

Proof. Define a function $\Phi : M_n(0) \to \prod_{i=1}^{n-1} G_{\text{mod}}(0)$, $\Phi(f) = (f_1, f_2, \cdots, f_{n-1})$, where for all $s \in \{1, 2, \cdots, n-1\}$, f_s is the branch action of the ultimately modular permutation f on the branch X_s. Each of the branch actions is a modular permutation on $\mathbb{Z}$, and furthermore they have zero flow. This is due to the following argument: Assume that f_s does not have zero flow for some $s \in \{1, 2, \cdots, n-1\}$, so $\phi_x(f_s) = p \neq 0$ for all $x \in \mathbb{Z} + \frac{1}{2}$. Let $x \in \mathbb{N} + \frac{1}{2}$ such that $x > M + N$, where

M is a bound for f, and N is a radius of non-modularity for f. Assume that $\phi_x^+(f_s) = q$, and so logically $\phi_x^-(f_s) = p - q$. Assume that $x_1, x_2, \cdots x_q$ are the integers that satisfy $x_i < x < f_s(x)$ for all $i \in \{1, 2, \cdots, q\}$ and that $y_1, y_2, \cdots y_{p-q}$ are the integers that satisfy $f_s(y_i) < x < y_i$ for all $i \in \{1, 2, \cdots, p-q\}$. Due to the choice of x, it means that any integer moved by f past x both starts and finishes on branch X_s, at a distance of more than N from the origin. This means that $\phi_{s,x}^o(f) = \phi_x^+(f_s) = q$ and $\phi_{s,x}^i(f) = \phi_x^-(f_s) = p - q$, hence $\phi_{s,x}(f) = p \neq 0$, which contradicts our assumption that f has zero flow. This argument shows that Φ is well-defined. We now show that Φ is a surjective homomorphism with kernel $FS(V_n)$.

In the proof of Lemma 3.3, we saw that if $f, g \in M_n$ with branch actions $f_1, f_2, \cdots, f_n$ for f and $g_1, g_2, \cdots, g_n$ for g, then $f \circ g \in M_n$ and has branch actions $f_1 \circ g_1, f_2 \circ g_2, \cdots, f_n \circ g_n$. It follows from this that

$$\begin{aligned}
\Phi(f \circ g) &= (f_1 \circ g_1, f_2 \circ g_2, \cdots f_{n-1} \circ g_{n-1}) \\
&= (f_1, f_2, \cdots, f_{n-1})(g_1, g_2, \cdots, g_{n-1}) \\
&= \Phi(f)\Phi(g)
\end{aligned}$$

so Φ is a homomorphism. Notice that $f \in FS(V_n)$ if, and only if, the branch actions of f are all $id_{\mathbb{Z}}$, which happens if, and only if, $\Phi(f) = (id_{\mathbb{Z}}, id_{\mathbb{Z}}, \cdots, id_{\mathbb{Z}})$, which happens iff $f \in \text{Ker } \Phi$. Hence $FS(V_n) = \text{Ker } \Phi$. It remains to show that Φ is surjective; to do this we need to define a collection of maps ψ_i for $i \in \{1, 2, \cdots, n-1\}$, that convert a modular permutation of $\mathbb{Z}$ into a modular permutation of V_n. Define $\psi_i : G_{\text{mod}} \to M_n$, where $\psi_i(f) : V_n \to V_n$ is the following map:

$$\psi_i(f)(v_{r,z}) = \begin{cases} v_{r,z} & r \neq i, n \\ v_{i,f(z)} & r = i \\ v_{n,-f(-z)} & r = n \end{cases}$$

Example 4.1. *Let $n = 5$, and let*

$$f = \cdots (-5, -3, -4)(-2, 0, 1)(1, 3, 2)(4, 6, 5) \cdots$$

Then

$$\psi_1(f) = \cdots (v_{n,5}, v_{n,3}, v_{n,4})(v_{n,2}, O, v_{n,1})(v_{1,1}, v_{1,3}, v_{1,2})(v_{1,4}, v_{1,6}, v_{1,5}) \cdots$$

and

$$\psi_4(f) = \cdots (v_{n,5}, v_{n,3}, v_{n,4})(v_{n,2}, O, v_{n,1})(v_{4,1}, v_{4,3}, v_{4,2})(v_{4,4}, v_{4,6}, v_{4,5}) \cdots$$

The maps ψ_i are well-defined (i.e. $\psi_i(f) \in M_n$ for all $f \in G_{\text{mod}}$): the branch action of $\psi_i(f)$ on any branch except X_i or X_n is clearly $id_{\mathbb{Z}}$. Its straightforward to see that the branch action of ψ_i on X_i is f, and furthermore the branch action of $\psi_i(f)$ on X_n is the reflection of f about 0, i.e. the branch action is f^τ, where $\tau : \mathbb{Z} \to \mathbb{Z}$, $\tau(z) = -z$. This implies that $\Phi(\psi_i(f_i)) = (id_{\mathbb{Z}}, \cdots, id_{\mathbb{Z}}, f_i, id_{\mathbb{Z}}, \cdots, id_{\mathbb{Z}})$. Let $(f_1, f_2, \cdots f_{n-1}) \in \prod_{i=1}^{n-1} G_{\text{mod}}(0)$. Along with the fact that Φ is a homomorphism, we see that

$$\begin{aligned}
\Phi(\psi_1(f_1) \circ \psi_2(f_2) \circ \cdots \circ \psi_{n-1}(f_{n-1})) &= \Phi(\psi_1(f_1))\Phi(\psi_2(f_2)) \cdots \Phi(\psi_{n-1}(f_{n-1})) \\
&= (f_1, id_{\mathbb{Z}}, \cdots, id_{\mathbb{Z}})(id_{\mathbb{Z}}, f_2, id_{\mathbb{Z}}, \cdots, id_{\mathbb{Z}}) \\
&\quad\ \cdots (id_{\mathbb{Z}}, \cdots, id_{\mathbb{Z}}, f_{n-1}) \\
&= (f_1, f_2, \cdots f_{n-1})
\end{aligned}$$

and so Φ is surjective. Hence Φ is a surjective homomorphism with kernel $FS(V_n)$, and so by the First Isomorphism Theorem we have $(M_n)_0/FS(V_n) \cong \prod_{i=1}^{n-1} G_{\text{mod}}(0)$. □

Recall that Theorem 2.3 says that $G_{\text{mod}}(0)$ is simple, and so by Theorem 4.2 $M_n(0)$ is a direct product of isomorphic simple groups. Proposition 1.1 implies that $M_n(0)$ is a characteristically simple group. So although $M_n(0)$ is not simple, it is an extension of a finite group by a characteristically simple group.

Examining the proof of Theorem 4.2 we can see that the result also holds for $n = 2$. Recall that $M_2 = ME$, the group of modular permutations of $\mathbb{Z}$ with modular ends, we have

Corollary 4.1. $ME(0)/FS(\mathbb{Z}) \cong G_{\text{mod}}(0)$

Thus $ME(0)$ is an extension of a finitary group by a simple group.

4.3 Decisive permutations

In this section, we introduce a generalisation of a finite flow permutation of V_n, called a **decisive permutation**. These permutations form a subgroup of $\mathrm{Sym}(V_n)$, which turns out to be the normalizer of the group of finite flow permutations of V_n in the full symmetric group. As usual, let Γ_n be a labelled n-branched star, and denote its vertex set by V_n, and its branches by $X_1, X_2, \cdots, X_n$. Recall that we denote the group of finite flow permutations of V_n by FF_n.

Definition 4.4. *Let $f \in Sym(V_n)$ and $r, s \in \{1, 2, \cdots, n\}$. We say that f* **sends** *branch X_r to branch X_s if $f(v) \in V(X_s)$ for all but finitely many $v \in V(X_r)$. We say that f is* **decisive** *if for all $r \in \{1, 2, \cdots, n\}$ there exists $s \in \{1, 2, \cdots, n\}$ such that f sends X_r to X_s.*

Example 4.2. *Let $n = 3$. The permutation*

$$f : V_n \to V_n,\ f(v) = \begin{cases} v_{2,x} & v = v_{1,x},\ x > 5 \\ v_{1,x} & v = v_{2,x},\ x > 5 \\ v & \textit{otherwise} \end{cases}$$

is decisive.

Notice that if f has finite flow on Γ_n then f is decisive, because each branch is sent to itself. However a decisive permutation need not have finite flow on G, as illustrated by Example 4.2.

Definition 4.5. *Let $f \in Sym(V_n)$. We say that f is a* **branch transposition** *if $\exists r, s \in \{1, 2, \cdots n\}$ such that*

$$f(v) = \begin{cases} v_{s,x} & v = v_{r,x}\ ,\ x \in \mathbb{N} \\ v_{r,x} & v = v_{s,x}\ ,\ x \in \mathbb{N} \\ v & \textit{otherwise.} \end{cases}$$

We define P_n to be the group generated by the branch transpositions, and call the elements of P_n the **branch permutations***.*

Clearly all the branch permutations are decisive, because X_r is sent to X_s and vice versa, and all other branches are sent to themselves. As the name suggests we can think of the elements as permuting the branches of Γ_n, and it can be easily shown that P_n is isomorphic to S_n. We shall now see that the elements of P_n are the only 'complicated' decisive maps.

Lemma 4.3. *Let $f \in Sym(V_n)$. Then f is decisive if, and only if, $f = g\sigma$, where g is a branch permutation and $\sigma \in FF_n$.*

Proof. Firstly assume $f = g\sigma$, where g is a branch permutation and $\sigma \in FF_n$. Without loss of generality we can assume that g is a branch transposition, since the composition of decisive maps is decisive. Assume g transposes X_r and X_s, for some $r, s \in \{1, 2, \cdots, n\}$. Now σ has finite flow on Γ_n, so in particular $|\sigma(V(X_r)) - V(X_r)| < \infty$. In other words $\sigma(v) \in V(X_r)$ for all but finitely many $v \in V(X_r)$. Therefore $g\sigma(v) \in V(X_s)$ for all but finitely many $v \in V(X_r)$. Similarly we have that $g\sigma(v) \in V(X_r)$ for all but finitely many $v \in V(X_s)$. Finally for $t \in \{1, 2, \cdots n\}$, $t \neq r, s$ we have that $g\sigma(v) \in V(X_t)$ for all but finitely many $v \in V(X_t)$, since g fixes all the vertices of X_t. Hence $g\sigma = f$ is decisive.

Conversely, assume that f is decisive. Define a map

$$\psi : \{1, 2, \cdots, n\} \to \{1, 2, \cdots, n\}$$

such that $\psi(r) = s$ if, and only if, f sends X_r to X_s. We show that this map is bijective, by showing that it is surjective. If ψ were not surjective then there exists $s \in \{1, 2, \cdots, n\}$ such that no $r \in \{1, 2, \cdots, n\}$ satisfies $s = f(r)$. In other words, no branch is sent to X_s by f (in particular X_s is not sent to itself). Since f is decisive, $f(v) \in V(X_s)$ for only finitely many $v \in V_n$. But f is a bijection, so we have a contradiction. Hence ψ is surjective and thus bijective.

Let g be the branch permutation which maps X_r toX_s if, and only if, f sends X_r to X_s. By the above argument f permutes the branches of Γ_n, modulo a finite number of vertices. We therefore construct σ as follows: if v satisfies $f(v) = v_{s,x}$ for some $x \in \mathbb{N}$, define $\sigma(v) = v_{r,x}$. If $f(v) = O$, then define $\sigma(v) = O$. σ is a well-defined map on V_n and we shall now see that $f = g\sigma$. If v satisfies $f(v) = v_{s,x}$, then

$$g(\sigma(v)) = g(v_{r,x}) = v_{s,x} = f(v)$$

and if $f(v) = O$, then

$$g(\sigma(v)) = g(O) = O = f(v)$$

since g is a branch permutation.

To complete the proof we need to show that σ is a bijection and has finite flow on Γ_n. It is clear that σ is a bijection because we know that $f = g\sigma$ and that f and g are bijections. Now let $r \in \{1, 2, \cdots n\}$. Since f is decisive, $\exists s \in \{1, 2, \cdots, n\}$ such that $f(v) \in V(X_s)$, for all but finitely many $v \in V(X_r)$. The choice of g implies that $\sigma(v) = (g^{-1}f)(v) \in$ for all but finitely many $v \in V(X_r)$. The choice of r was arbitrary, and so σ has finite flow on Γ_n. $\square$

Let N denote $N_{Sym(V_n)}(FF_n)$, i.e. the normaliser of the group of finite flow permutations of V_n in the full symmetric group. We now show that the decisive permutations are exactly the elements of N.

Lemma 4.4. *Let $f \in Sym(V_n)$. If f is decisive, then $f \in N$.*

Proof. Let f be decisive. Lemma 4.3 tells us that $f = g\sigma$ where g is a branch permutation and $\sigma \in FF_n$. We will show that $g \in N$. Let $s \in \{1, 2, \cdots, n\}$. By the argument in Lemma 4.3, there exists $r \in \{1, 2, \cdots, n\}$ such that g maps X_r to X_s. Let $\tau \in FF_n$ and consider τ^g, the conjugate of τ by g: We claim that $\tau^g(v) \in V(X_s)$ holds for all but finitely many $v \in V(X_s)$.

The condition $g^{-1}(v) \in V(X_r)$ holds for all $v \in V(X_s)$ because g maps X_r to X_s. Hence the condition $\tau g^{-1}(v) \in V(X_r)$ holds for all but finitely many $v \in V(X_s)$ because τ has finite flow on Γ_n. Hence the condition $g\tau g^{-1}(v) \in V(X_s)$ holds for all but finitely many $v \in V(X_s)$ because g maps X_r to X_s, and so the claim is true.

The branch X_s was chosen arbitrarily, and so it follows from the definition that $\tau^g \in FF_n$, and so $g \in N$. As $\sigma \in FF_n$ it follows that $\sigma \in N$ and so we have that $f \in N$. $\square$

In proving the next lemma, it will be useful to have a way of ordering subsets of a branch of Γ_n. Let $r \in \{1, 2, \cdots, n\}$, and let A be a subset of $V(X_r)$. Order the elements of A according to their distance from the origin; so a_1 is the element of A which is the shortest distance away from the origin, a_2 the element of A

which is the second shortest distance away from the origin, and so on. We call this ordering the **natural ordering** of the set A. There is no ambiguity in this ordering, as we are only ordering elements of one particular branch of Γ_n. Now we can proceed with the next result.

Lemma 4.5. *Let f be a permutation of V_n. If $f \in N$, then f is decisive.*

Proof. We prove that if f is not decisive, then $f \notin N$. Assume that f is not decisive. This means that there is $r \in \{1, 2, \cdots, n\}$ such that the branch X_r is not sent to any branch of Γ_n by f. In particular, X_r is not sent to itself, and so

$$|\{v \in X_r : f(v) \notin X_r\}| = \infty \tag{4.2}$$

In other words, f moves infinitely many elements of X_r away from that branch. Let $R = \{v \in X_r : f(v) \in X_r\}$, the set of elements of X_r that are not moved to another branch of Γ_n. We distinguish two cases: R is infinite, and R is finite. Although we shall work with two separate cases, the overall method is the same: construct a finite flow permutation of V_n such that when we conjugate it by f, we no longer have a finite flow permutation.

Firstly, consider the case where R is infinite. Condition (4.2) tells us that f moves infinitely many elements of X_r away from that branch, but as there are only a finite number of branches of Γ_n, there is $s \in \{1, 2, \cdots, n\}$ distinct from r such that f moves infinitely many elements of X_r onto X_s. In other words, there is $s \in \{1, 2, \cdots, n\}$ distinct from r such that $\{v \in X_r : f(v) \in X_s\}$ is infinite. Let $S = \{v \in X_r : f(v) \in X_s\}$. Order the sets R and S with the natural ordering, so $R = \{r_1, r_2, \cdots\}$ and $S = \{s_1, s_2, \cdots\}$ where the elements are indexed by increasing distance from the origin. Now we define the key permutation: let $\sigma : V_n \to V_n$,

$$\sigma = (\cdots, s_3, r_2, s_2, r_1, s_1, v_{s,1}, v_{s,2}, v_{s,3}, \cdots)$$

The permutation σ has finite flow because σ moves at most one vertex between any two branches of Γ_n. Furthermore, for any $v \in f(R)$,

$$\begin{aligned}\sigma^f(v) &= f\sigma f^{-1}(v)\\ &= f\sigma(r_i) \qquad \text{for some } i \in \mathbb{N}\\ &= f(s_i)\\ &\in f(S)\end{aligned}$$

Now $f(R)$ is infinite and is a subset of X_r, but $f(S)$ is a subset of X_s. Therefore σ^f moves infinitely many vertices of X_r onto X_s, and so σ^f does not have finite flow. Hence $f \notin N$.

Now consider the case where R is finite. As in the case where R is infinite, condition (4.2) implies there is $s \in \{1, 2, \cdots, n\}$ distinct from r such that f moves infinitely many elements of X_r onto X_s; in other words, there is $s \in \{1, 2, \cdots, n\}$ distinct from r such that $\{v \in X_r : f(v) \in X_s\}$ is infinite. However, if X_s cannot be the only branch that f moves infinitely many elements of X_r onto, because if it were then since R is finite, f sends X_r to X_s which contradicts our original assumption that X_r is not sent to any branch by f. Therefore there is $t \in \{1, 2, \cdots, n\}$, distinct from r and s, such that f moves infinitely many elements of X_r onto X_t; i.e. $\{v \in X_r : f(v) \in X_t\}$ is infinite. Let $S = \{v \in X_r : f(v) \in X_s\}$ and $T = \{v \in X_r : f(v) \in X_t\}$, and order these sets with the natural ordering. Now we can define the key permutation: let $\sigma : V_n \to V_n$,

$$\sigma = (\cdots, s_3, t_2, s_2, t_1, s_1, v_{s,1}, v_{s,2}, v_{s,3}, \cdots)$$

The permutation σ has finite flow because σ moves at most one vertex between any two branches of Γ_n. Furthermore, for any $v \in f(T)$,

$$\begin{aligned}\sigma^f(v) &= f\sigma f^{-1}(v)\\ &= f\sigma(t_i) \qquad \text{for some } i \in \mathbb{N}\\ &= f(s_i)\\ &\in f(S)\end{aligned}$$

Now $f(T)$ is infinite and is a subset of X_t, but $f(S)$ is a subset of X_s. Therefore σ^f moves infinitely many vertices of X_t onto X_s, and so σ^f does not have finite flow. Hence $f \notin N$.

In both of the possible cases we have shown that if f is not decisive, then $f \notin N$, therefore the result is proved. □

Corollary 4.2. *Let $f \in Sym(V_n)$. Then f is decisive if, and only if, $f \in N$.*

Proof. Lemmas 4.4 and 4.5. □

Theorem 4.3. *Let Γ_n be a labelled n-branched star for some $n \in \mathbb{N}$, and let V_n denote the vertex set of Γ_n. Then*

$$N_{\mathrm{Sym(V_n)}}(FF_n)/FF_n \cong S_n$$

Proof. Corollary 4.2 tells us that N is exactly the set of decisive permutations. Lemma 4.3 tells us that if f is decisive then it can be written as a composition of a finite flow permutation and a branch permutation $g \in P_n$. Define a map $\chi : N \to P_n$, $\chi(f) = g$ if, and only if, g is the branch permutation specified in the proof of Lemma 4.3. It can be easily checked that this map is an epimorphism with kernel FF_n, and we have already noted that $P_n \cong S_n$. □

Chapter 5

Results about the cyclizer series of some interesting infinite permutation groups

In this chapter we investigate some more interesting infinite groups, and determine some information pertaining to their cyclizer series.

5.1 The cyclizer series of the cross group

Let Γ_4 be a labelled n-branched star. Recall from Chapter 3 that

$$\sigma_{r,s} = (\cdots, v_{r,2}, v_{r,1}, O, v_{s,1}, v_{s,2}, \cdots)$$

for all distinct $r, s \in \{1, 2, \cdots, n\}$, and that when $r \equiv s \bmod n$ we write σ_r instead.

Definition 5.1. *We define the* **cross group** *to be the permutation group of V_4 generated by σ_1 and σ_3. We denote the cross group by G_+.*

The cross group is so called because if we label the branches of Γ_4 in a certain way, the geometric interpretations of σ_1 and σ_3 form a cross. It is clear from the definition that $G_+ \leq G_4$, where G_4 is the permutation group on V_4 that was studied on Chapter 3, and so G_+ consists of ultimately elementary permutations that satisfy the Kirchoff property. However, there are structural differences between

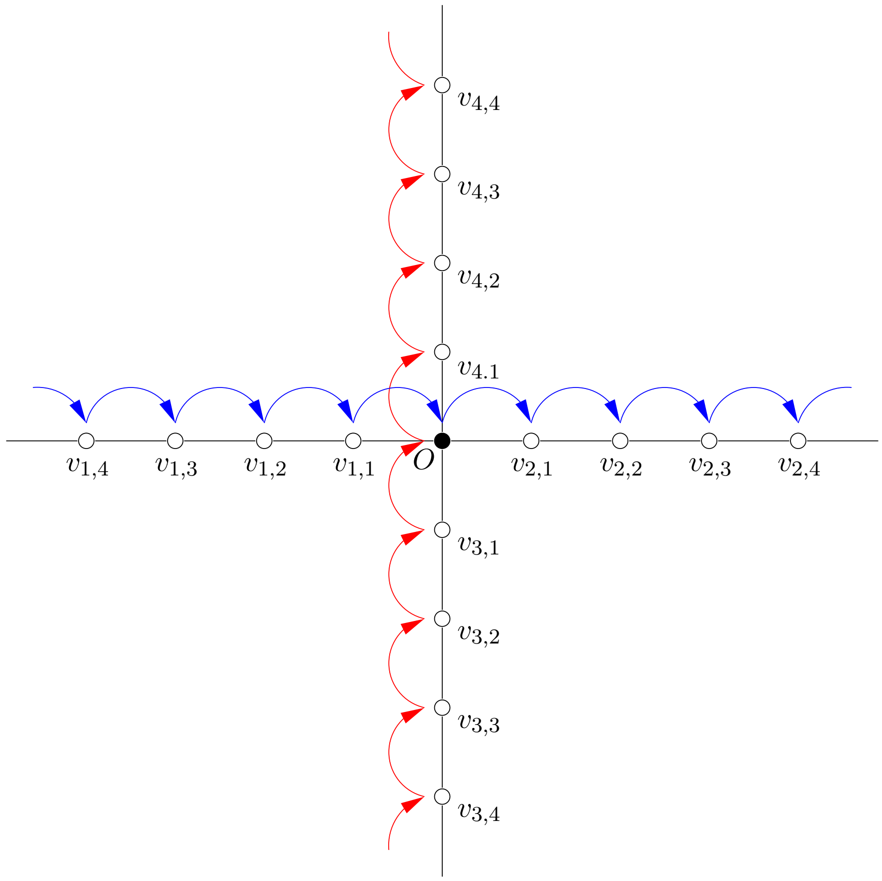

Figure 5-1: Geometric interpretation of the generators of the cross group G_+

these two groups; in particular, all the finitary elements of G_+ are alternating. This is unlike G_4, where a finitary element of G_4 can be any finitary permutation of V_4.

Lemma 5.1. *Let f be a finitary permutation of V_4 , and suppose that $f \in G_+$. Then $f \in Alt(V_4)$.*

Proof. A straightforward calculation shows that $[\sigma_1, \sigma_3] = (O, v_{1,1}, v_{4,1})$, and so $[\sigma_1, \sigma_3] \in \mathrm{Alt}(V_4)$. It follows that the derived subgroup G_+' is contained in $\mathrm{Alt}(V_4)$, since G_+' is generated by the commutators of G. Therefore, if $g \in G_+$ we can write

$$g = \sigma_1^{s_1} \sigma_3^{s_3} \varepsilon$$

for some $s_1, s_3 \in \mathbb{Z}$ and some $\varepsilon \in G_+'$, because G_+/G_+' is abelian and G_+ is generatedby σ_1 and σ_3. Due to the above observations about G'_+, it follows that ε is an alternating finitary permutation of V_4.

Now $f \in G_+$, so we can write $f = \sigma_1^{s_1}\sigma_3^{s_3}\varepsilon$ with s_1, s_3, ε as above. It is easy to see that for all $v \in X_1$ a sufficient distance away from the origin,

$$f(v) = \sigma_1^{s_1}(v)$$

because ε does not move v as ε is finitary, and powers of σ_3 do not move v by definition. Furthermore f is finitary and so $f(v) = v$ for all $v \in V_4$ a sufficient distance from the origin. Combining these two observations, we see that for all $v \in X_1$ a sufficient distance away from the origin,

$$\sigma_1^{s_1}(v) = v$$

Since $v \in \mathrm{supp}(\sigma_1)$, if follows that $s_1 = 0$. Similarly,

$$f(v) = \sigma_3^{s_3}(v) = v$$

for all $v \in X_3$ a sufficient distance from the origin, and so as $v \in \mathrm{supp}(\sigma_3)$, it follows that $s_3 = 0$. Hence $f = \varepsilon$, a finitary alternating permutation $\square$

What about the cyclizer series of G_+? Observe that both σ_2 and σ_4 are contained in $\mathrm{Cyc}(G_+)$: firstly,

$$\begin{aligned}\sigma_3^{-1}\sigma_1^{-1} &= (\cdots, v_{4,2}, v_{4,1}, O, v_{3,1}, v_{3,2}, \cdots)(\cdots, v_{2,2}, v_{2,1}, O, v_{1,1}, v_{1,2}, \cdots)\\ &= (\cdots, v_{4,2}, v_{4,1}, O, v_{1,1}, v_{1,2}, \cdots)(\cdots, v_{2,2}, v_{2,1}, v_{3,1}, v_{3,2}, \cdots)\\ &= \sigma_4(\cdots, v_{2,2}, v_{2,1}, v_{3,1}, v_{3,2}, \cdots)\end{aligned}$$

so σ_4 is involved in $\sigma_3^{-1}\sigma_1^{-1}$, and so $\sigma_4 \in \mathrm{Cyc}(G_+)$. Furthermore,

$$\begin{aligned}\sigma_1^{-1}\sigma_3^{-1} &= (\cdots, v_{2,2}, v_{2,1}, O, v_{1,1}, v_{1,2}, \cdots)(\cdots, v_{4,2}, v_{4,1}, O, v_{3,1}, v_{3,2}, \cdots)\\ &= (\cdots, v_{2,2}, v_{2,1}, O, v_{3,1}, v_{3,2}, \cdots)(\cdots v_{4,2}, v_{4,1}, v_{1,1}, v_{1,2}, \cdots)\\ &= \sigma_2(\cdots v_{4,2}, v_{4,1}, v_{1,1}, v_{1,2}, \cdots)\end{aligned}$$

so σ_2 is involved in $\sigma_1^{-1}\sigma_3^{-1}$, and so $\sigma_2 \in \mathrm{Cyc}(G_+)$. Hence all of the generators of

G_4 are contained in $\mathrm{Cyc}(G_+)$, and so $G_4 \leq \mathrm{Cyc}(G_+)$.

This fact implies that we have a series of inequalities

$$G_+ \leq G_4 \leq \mathrm{Cyc}(G_+) \leq \mathrm{Cyc}(G_4) \leq \mathrm{Cyc}^2(G_+) \leq \mathrm{Cyc}^2(G_4)$$

But we know from 3.3 that $\mathrm{Cyc}^2(G_4)$ is cycle-closed, i.e $\mathrm{Cyc}^2(G_4) = \mathrm{Cyc}(G_4)$. Due to the series of inequalities above this means that $\mathrm{Cyc}^2(G_+) = \mathrm{Cyc}(G_4)$, i.e.

Theorem 5.1. *$Cyc^2(G_+)$ is cycle-closed.*

Hence the cyclizer length of G_+ is at most 2.

5.2 The infinite ladder graph, and the cyclizer series of a group acting on it

In this section we investigate the cyclizer series of permutation group that arose naturally from a geometric point-of-view. As such, we will be drawing figures often and using them to make definitions, as they will generally be the simplest way to describe what is happening. The permutation group we are interested in is a group of permutations of a new type of graph. We now define this graph.

Definition 5.2. *Let S_1 and S_2 be disjoint 2-branched stars. Let L be the graph with vertex set $V(S_1) \cup V(S_2)$ and edge set as shown below in Figure 5-2. We call L the* **infinite ladder***.*

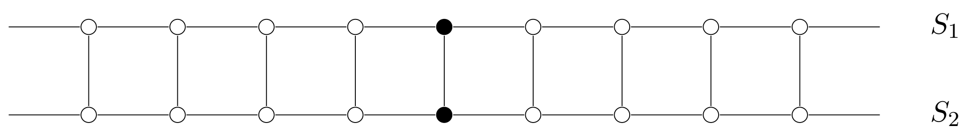

Figure 5-2: Illustration of the edge set of the graph L

In other words, the infinite ladder is the graph union of two distinct 2-branched stars, with an extra set of edges that connect the two 'pieces' in a natural way. Next we define the group of permutation of $V(L)$ that we are interested in.

Definition 5.3. *Let α be the permutation of L represented by the unbroken arrows in Figure 5-3, β be the permutation of L represented by the dashed arrows in Figure 5-3, and γ be the permutation of L represented by the dotted arrows in Figure 5-3. We define the* **ladder group** *to be the group generated by these three permutations, and we denote the ladder group by G_L.*

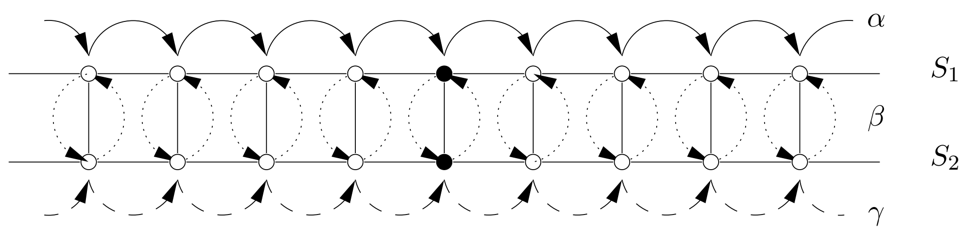

Figure 5-3: Fragment of the graph L with interpretations of the generators of G_L

In other words, α translates vertices on the 'top line' of L one place to the right; β translates vertices on the 'bottom line' of L one place to the right, and γ transposes adjacent elements on separate 'lines'. What is the cyclizer series of G_L? In order to help us understand what is going on in this group, we choose a useful labelling of L, which is illustrated in Figure 5-4 below. In essence, we

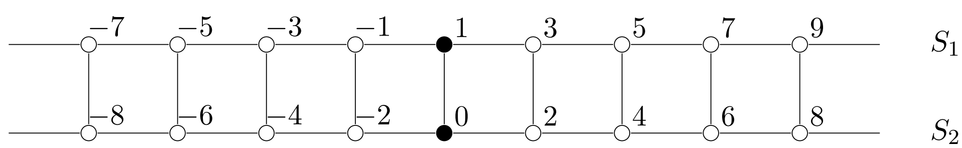

Figure 5-4: A useful labelling of the graph L

label the 'top line' with the odd integers, increasing as we head to the 'right', so that the origin of S_1 is labelled with the integer 1. Then we label the 'bottom line' with the even integers, increasing toward the 'right', so that the origin of S_2 is labelled with the integer 0. Due to this labelling, we may consider G_L to be a permutation group on $\mathbb{Z}$. In particular, under this labelling the generators of G_L

may be written as follows:

$$\alpha = (\cdots - 3, -1, 1, 3, \cdots)$$
$$\beta = (\cdots - 4, -2, 0, 2, 4, \cdots)$$
$$\gamma = \cdots (-2, -1)(0, 1)(2, 3) \cdots$$

Now we are in a position to understand the cyclizer of G_L. Firstly, we see that $FS(\mathbb{Z})$, the group of finitary symmetric permutations of $\mathbb{Z}$, is a subgroup of $\mathrm{Cyc}(G_L)$.

Lemma 5.2. *$FS(\mathbb{Z}) \leq Cyc(G_L)$*

Proof. The cycles involved in γ are precisely the transpositions $(2k, 2k+1)$ for all integers k. As γ is a generator of G_L, it follows that all of these cycles are contained in $\mathrm{Cyc}(G_L)$. By definition, the cyclizer of G_L contains all powers of α, and so for all $k, l \in \mathbb{Z}$ the transpositions $(2k, 2l+1)$ are contained in $\mathrm{Cyc}(G_L)$, because they can be written as the conjugate of $(2k, 2k+1)$ by α^{l-k}. Furthermore, for all $k, l \in \mathbb{Z}$ we have that $(2k, 2l) = (2k, 2l+1)^{(2l,2l+1)}$ and $(2k+1, 2l+1) = (2k, 2l+1)^{(2k,2k+1)}$, and so all the transpositions of the form $(2k, 2l)$ and $(2k+1, 2l+1)$ are also contained in $\mathrm{Cyc}(G_L)$. Every transposition of two integers will transpose either two odd integers, two even integers, or one odd integer and one even integer, hence every transposition of two integers is contained in $\mathrm{Cyc}(G_L)$. The group $FS(\mathbb{Z})$ is generated by the set of all transpositions of two integers, and so $FS(\mathbb{Z}) \leq \mathrm{Cyc}(G_L)$. □

Next, observe that G_{mod}, the group of modular permutations of $\mathbb{Z}$, is a subgroup of $\mathrm{Cyc}(G_L)$.

Lemma 5.3. *$G_{\mathrm{mod}} \leq Cyc(G_L)$*

Proof. A straightforward calculations shows that $\gamma\alpha = (\cdots, -2, -1, 0, 1, 2, \cdots)$, and since the group G_2 is generated by $(\cdots, -2, -1, 0, 1, 2, \cdots)$ it follows that G_2 is a subgroup of G_L. Therefore $\mathrm{Cyc}(G_2)$ is a subgroup of $\mathrm{Cyc}(G_L)$. Theorem 1.4 tells us that $\mathrm{Cyc}(G_2)$ is the group G_{mod}, and so we are done. □

Corollary 5.1. *$Cyc^2(G_2) \leq Cyc(G_L)$*

Proof. Theorem 1.4 tells us that $\mathrm{Cyc}^2(G_2)$ is generated by G_{mod} and $FS(\mathbb{Z})$. Both of these groups are subgroups of $\mathrm{Cyc}(G_L)$ by Lemma 5.2 and Lemma 5.3 respectively, and therefore $\mathrm{Cyc}^2(G_2)$ is also a subgroup of $\mathrm{Cyc}(G_L)$. □

A natural question would be: can the inequality in Corollary 5.1 be made into an equality? The answer to this is yes.

Lemma 5.4. *$Cyc(G_L) \leq Cyc^2(G_2)$*

Proof. It is easy to check that α, β and γ are modular permutations with modulus 1, and since these three permutations generate G_L, it follows that G_L is a subgroup of G_{mod}. Theorem 1.4 implies that $G_{\mathrm{mod}} = \mathrm{Cyc}(G_2)$, and so G_L is a subgroup of $\mathrm{Cyc}(G_2)$. It follows that the cyclizer of G_L is a subgroup of the cyclizer of $\mathrm{Cyc}(G_2)$, and so we have the result. □

Putting this all together, we have:

Theorem 5.2. *$Cyc(G_L) = Cyc^2(G_2)$*

Theorem 1.4 tells us that the cyclizer length of G_2 is 3, and so the cyclizer length of G_L is 2.

Chapter 6

Concluding Remarks

There are plenty of open questions that can be addressed regarding the topics touched upon in this work. The cyclizer series of the infinite cyclic group acting naturally on itself has been studied at length. Since the infinite cyclic group can be viewed as a free abelian group of rank 1, a sensible area for further work would be to investigate the cyclizer series of a free abelian group of rank n. In fact, we can see that the cross group from Chapter 5 is inherently related to the cyclizer series of the free abelian group of rank n.

If we consider the free abelian group of rank n as the direct product of n copies of the integers, then it can be shown that the cycles involved in elements of the induced permutation group (in other words, the group $\mathbb{Z}^n$, take the form

$$(\cdots, \mathbf{a} - \mathbf{2z}, \mathbf{a} - \mathbf{z}, \mathbf{a}, \mathbf{a} + \mathbf{z}, \mathbf{a} + \mathbf{2z}, \cdots)$$

for all $\mathbf{a}, \mathbf{z} \in \mathbb{Z}^n$. The cyclizer of $\mathbb{Z}^n$ is generated in part by these cycles, so we are obviously interested how these cycles act under composition. If we take two cycles of the above form, then there is a natural bijection from the union of their support onto the vertex set of a 4-branched star, such that the two cycles are essentially the two generators of the cross group. This motivates further study into the cross group; in particular finding a normal form for elements of the cyclizer of the cross group would be very useful, as then we would have most of the necessary information to understand how cycles involved in elements of $\mathbb{Z}^n$ interact. This observation also motivates the study of a 'generalised cross group', where the underlying graph is a 2 branched star with multiple 2-branched stars crossing perpendicularly, rather than just one extra 2-branched star. Studying this natural permutation group of the vertices of this graph would give us information about how any number of cycles involved in the free abelian group of rank n interact. In Chapter 5 we saw that the cyclizer length of the cross group is at most 2: Is it the case that the cylizer length is exactly 2?. Is the cyclizer length of the 'generalised' cross group' also 2?

The other group studied in Chapter 5, the group G_L acting on the infinite ladder L, is a good area for further work. Firstly, if we define a 'generalised infinite ladder', where instead of taking the union of two copies of a 2-branched star with a particular edge set, we take instead the union of n copies of a 2-branched with the natural generalisation of the edge set (i.e we 'stack' the branches horizontally and connect them by all possible vertical edges), then it can be shown that the cyclizer series of the natural generalisation of G_L has an *identical* cyclizer series to G_L. This is achieved by labelling the vertices of the 'generalised infinite ladder' by partitioning the integers into their equivalence classes modulo n (as a generalisation of partitioning into even and odd integers when labelling G_L), and labelling each of the horizontal pieces with one of the equivalence classes in the natural way. The labelling gives a tidy way of interpreting the generators of the generalised group, and some fairly standard calculations finish the job.

A further observation can be made about G_L and its proposed generalisations: although we constructed these groups from a purely geometrical point-of-view, it is easy to see that G_L is the permutation group induced by the wreath product $C_\infty \; wr \; S_2$, and that the generalised groups are the permutation groups induced by the wreath products $C_\infty \; wr \; S_n$. Therefore the cyclizer series of $C_\infty \; wr \; S_n$ is *independent* of n, and the cyclizer length of all of these wreath products is 2, which is certainly interesting. Wreath products often have very nice representations as permutations of the vertices of a graph, and so investigating the cyclizer series of infinite wreath products seems to be a pertinent area for further research. Perhaps it can be shown that the cyclizer series of any infinite wreath product, where the complement group is S_n for some $n \in \mathbb{N}$, is independent of n.

In Cameron's original paper, he conjectured that the maximal cyclizer length for all groups might be 3. Proving this conjecture, or finding a counter-example seems to be an excellent topic of research, as so far almost all the theory for the cyclizer series of an infinite permutation group has been focused on investigating specific groups. The author feels that if this conjecture turns out to be true, then this might be a very challenging problem to solve, because there is so little understood about infinite groups in general.

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